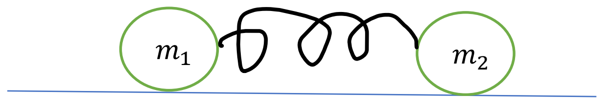
CM1: Two particles of mass m_1 and m_2 slide freely on a horizontal frictionless track and are connected by a spring with a force constant k. Derive the Lagrange equations of motion and find the oscillation frequency.



Solution 1: CM and Rel. Motion

The kinetic and potential energy are, respectively:

$$T = \frac{1}{2} m_1 \, \dot{x}_1^2 + \frac{1}{2} m_2 \, \dot{x}_2^2$$

$$V = \frac{1}{2} k (x_2 - x_1)^2$$

Thus, the Lagrangian is:

$$L = T - V$$
$$L = \frac{1}{2} m_1 \, \dot{x}_1^2 + \frac{1}{2} m_2 \, \dot{x}_2^2 - \frac{1}{2} k (x_2 - x_1)^2$$

Use the Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \, \dot{x}_1}\right) = \frac{\partial L}{\partial x_1}$$
$$\frac{d}{dt}(m_1 \, \dot{x}_1 + 0 - 0) = \frac{\partial}{\partial x_1}\left(\frac{1}{2} m_1 \, \dot{x}_1^2 + \frac{1}{2} m_2 \, \dot{x}_2^2 - \frac{1}{2} k (x_2 - x_1)^2\right)$$
$$\frac{d}{dt}(m_1 \, \dot{x}_1 + 0 - 0) = 0 + 0 + \frac{\partial}{\partial x_1}\left(-\frac{1}{2} k (x_2 - x_1)^2\right)$$

$$m_1 \ddot{x}_1 = \, -\frac{1}{2} k \, (-2)(x_2 - x_1)$$

Thus,

$$m_1 \ddot{x}_1 = \, k(x_2 - x_1)$$

The E-L equation for x_2 is the same except on the right-hand side we pick up a negative sign:

$$\frac{\partial}{\partial x_2}\left(-\frac{1}{2} k (x_2 - x_1)^2\right) = -\frac{1}{2} k \, (2)(x_2 - x_1) = k(x_2 - x_1)$$

Therefore, the two equations of motion are:

$$m_1 \ddot{x}_1 = \, k(x_2 - x_1) \qquad (1)$$

$$m_2 \ddot{x}_2 = \, -k(x_2 - x_1) \qquad (2)$$

This matches with what we would get if we had used $F = ma$ and Newton's third law. To solve Eq. (1) and (2), it's convenient to go to the center-of mass and relative co-ordinates:

$$r_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \qquad r_{rel} = x_2 - x_1$$

If we add Eq.(1) and (2),

$$m_1 \ddot{x}_1 + m_2 \ddot{x}_2 = 0$$

Divide by $m_1 + m_2$:

$$\frac{m_1 \ddot{x}_1 + m_2 \ddot{x}_2}{m_1 + m_2} = 0$$

Looking at r_{cm} we see that this means:

$$\ddot{r}_{cm} = 0$$

Thus, the center-of-mass moves with constant velocity.

Sidenote: this is true in general for isolated systems, since if you sum over all $F = ma$, like we did here, you'll get cancellations of all the forces by the action-reaction pairs with for the Force sides and for the other the total mass times the acceleration of the center of mass. The potential usually depends on the distances between two objects, $V(x_1, x_2) = V(x_2 - x_1)$, and thus the dynamics is usually associated with relative positions. This was the motivation for choosing these two co-ordinates.

Now,

$$\ddot{r}_{rel} = \ddot{x}_2 - \ddot{x}_1$$

Divide both sides of Eq. (1) and (2) by their respective masses:

$$\ddot{x}_1 = \frac{k}{m_1}(x_2 - x_1) \tag{3}$$

$$\ddot{x}_2 = -\frac{k}{m_2}(x_2 - x_1) \tag{4}$$

Subtract Eq (3) from (4)

$$\ddot{x}_2 - \ddot{x}_1 = -\frac{k}{m_2}(x_2 - x_1) - \frac{k}{m_1}(x_2 - x_1) \tag{5}$$

Using the definition of r_{rel} in Eq. (5):

$$\ddot{r}_{rel} = -\frac{k}{m_2} r_{rel} - \frac{k}{m_1} r_{rel}$$

Thus,

$$\ddot{r}_{rel} = -k\left(\frac{1}{m_1}+\frac{1}{m_2}\right)r_{rel}$$

The differential equation for simple harmonic motion is:

$$\ddot{z} = -\omega^2 z$$

We then have:

$$\boxed{\omega = \sqrt{k\left(\frac{1}{m_1}+\frac{1}{m_2}\right)}}$$

Solution 2: Getting Eigenvalues

Up to Eq. (1) and (2) the procedure is the same.

Now, let's use matrix notation to re-express Eq. (3) and (4):

$$\begin{pmatrix}\ddot{x}_1\\ \ddot{x}_2\end{pmatrix} = \begin{pmatrix}-\frac{k}{m_1} & \frac{k}{m_1}\\ \frac{k}{m_2} & -\frac{k}{m_2}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} \tag{6}$$

Let

$$W = \begin{pmatrix}-\frac{k}{m_1} & \frac{k}{m_1}\\ \frac{k}{m_2} & -\frac{k}{m_2}\end{pmatrix}$$

Diagonalizing W can help us solve the equations. Why? If $W = O^{-1}DO$ where D is a diagonal matrix and we let:

$$\boldsymbol{r} = \begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

then Eq. (6) can be expressed as:

$$\ddot{\boldsymbol{r}} = W\boldsymbol{r}$$
$$\ddot{\boldsymbol{r}} = O^{-1}DO\boldsymbol{r} \quad (multily\ by\ O\ both\ sides)$$

$$O\ddot{\boldsymbol{r}} = OO^{-1}DO\boldsymbol{r}$$

$$O\ddot{\boldsymbol{r}} = DO\boldsymbol{r}$$

If we let $\boldsymbol{y} = O\boldsymbol{r}$ then:

$$\ddot{\boldsymbol{y}} = Dy$$

Since D is a diagonal matrix of the form:

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Then:

$$\ddot{y_1} = \lambda_1 y_1$$

$$\ddot{y_2} = \lambda_2 y_2$$

Diagonalizing thus decouples the two components. Simple harmonic motion is of the form:

$$\ddot{y} = -\omega^2 \, y$$

Thus,

$$\omega_{1,2} = \sqrt{-\lambda_{1,2}}.$$

Note: if we have oscillatory motion the eigenvalue should be negative, hence the value under the square root should be positive.

Now, we diagonalize W:

$$\det(W - \lambda I) = 0$$

$$\begin{vmatrix} -\dfrac{k}{m_1} - \lambda & \dfrac{k}{m_1} \\ \dfrac{k}{m_2} & -\dfrac{k}{m_2} - \lambda \end{vmatrix} = 0$$

$$\left(-\frac{k}{m_1} - \lambda\right)\left(-\frac{k}{m_2} - \lambda\right) - \frac{k}{m_1}\frac{k}{m_2} = 0$$

$$\left(\frac{k}{m_1} + \lambda\right)\left(\frac{k}{m_2} + \lambda\right) - \frac{k}{m_1}\frac{k}{m_2} = 0$$

$$\frac{k}{m_1}\frac{k}{m_2} + \frac{k}{m_1}\lambda + \frac{k}{m_2}\lambda + \lambda^2 - \frac{k}{m_1}\frac{k}{m_2} = 0$$

$$\frac{k}{m_1}\lambda + \frac{k}{m_2}\lambda + \lambda^2 = 0$$

$$\left(\frac{k}{m_1} + \frac{k}{m_2} + \lambda\right)\lambda = 0$$

We have two eigenvalues:

$$\lambda = 0, -\left(\frac{k}{m_1} + \frac{k}{m_2}\right)$$

The first corresponds to the center-of-mass motion, which is constant. The second is the oscillations frequency and is given by $\omega = \sqrt{-\lambda}$:

$$\boxed{\omega = \sqrt{k\left(\frac{1}{m_1} + \frac{1}{m_2}\right)}}$$

This method of finding the eigenvalues of a matrix is used when looking for the frequencies of the normal modes in general.