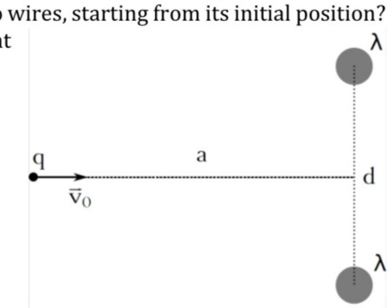
2021 EM 2

EM2: We consider the setup in the figure below. We have two infinite wires perpendicular to the plane of this sheet of paper, each with uniform linear charge density λ. The distance between the wires is d. We also have a positive point charge at a distance "a" along the perpendicular bisector of the line connecting the two wires. This point charge has mass m, charge $q > 0$ and an initial velocity v_0 along the perpendicular bisector towards the wires.

a) What is the magnitude and direction of the electrical force on the point charge as a function of the distance to the line connecting the two wires?
b) What is the work done by the electrical forces on the point charge as a function of the distance to the line connecting the two wires, starting from its initial position?
c) How large should v_0 minimally be so that the point charge passes the two wires?



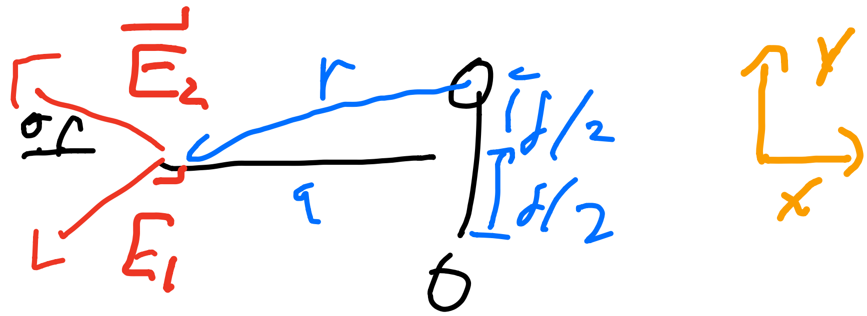
For an infinite wire

Gauss' law

$$\oint \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$$

$$E(2\pi r \cancel{l}) = \frac{\lambda \cancel{l}}{\varepsilon_0}$$

$$E = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{r}$$



$$E_x = E_{1,x} + E_{2x}$$

$$= 2E_{1,x} \quad \text{(By symmetry)}$$

$$= -2\,\frac{\lambda}{2\pi\varepsilon_0}\,\frac{1}{r}\,\cos\theta$$

$$= -\frac{\lambda}{\pi\varepsilon_0}\,\frac{1}{r}\,\frac{a}{r}$$

$$= -\frac{\lambda}{\pi\varepsilon_0}\,\frac{a}{r^2}$$

$$E_x = -\frac{\lambda}{\pi\varepsilon_0}\,\frac{a}{a^2+d^2/4}$$

Since $\vec{F} = q\vec{E}$

$$\boxed{\vec{F} = -\frac{q\lambda}{\pi\varepsilon_0}\frac{a}{a^2+d^2/4}\hat{x}}$$

b)

$$W = \int_C \vec{F}\cdot d\vec{l}$$

$$W = \int_a^0 dx\left(\frac{q\lambda}{\pi\varepsilon_0}\frac{x}{x^2+d^2/4}\right)$$

$$W = \frac{q\lambda}{\pi\varepsilon_0}\int_a^0 dx\,\frac{x}{x^2+d^2/4}$$

flip integration order

$$= -\frac{q\lambda}{\pi\varepsilon_0}\int_0^a dx\,\frac{x}{x^2+d^2/4}$$

$$W = -\frac{q\lambda}{\pi\varepsilon_0}\left[\ln\left(x^2+d^2/4\right)\Big|_0^a\right]$$

$$W = \frac{-q\lambda}{2\pi\varepsilon_0} \ln\left(\frac{a^2 + d^2/4}{d^2/4}\right)$$

$$\boxed{W = \frac{-q\lambda}{2\pi\varepsilon_0} \ln\left(1 + \frac{4a^2}{d^2}\right)}$$

Note: W must be negative since $\vec{F}$ and $d\vec{x}$ anti-aligned.

c) By work-energy theorem,

$$W = \Delta KE$$

$$W = 0 - \frac{1}{2} m v_0^2$$

$$-\frac{q\lambda}{\pi\varepsilon_0}\ln\left(1+\frac{4a^2}{d^2}\right) = -\frac{1}{2}mv_0^2$$

$$v_0^2 = \frac{2q\lambda}{m\pi\varepsilon_0}\left[\ln\left(1+\frac{4a^2}{d^2}\right)\right]$$

$$\boxed{v_0 = \sqrt{\frac{2q\lambda}{m\pi\varepsilon_0}\ln\left(1+\frac{4a^2}{d^2}\right)}}$$