QM2: Find the energy eigenstates and eigenvalues of a particle confined to a delta function potential $V(x) = -\beta\delta(x)$, where β is a positive real constant. You may find the following definition of Dirac delta function $\delta(x)$ useful:

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$$

The Schroedinger equation is

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) + V(x)\psi(x) = E\psi(x) \quad (1)$$

Given $V(x) = -\beta\delta(x)$

For $x \neq 0$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) = E\psi(x)$$

$$\frac{\partial^2}{\partial x^2}\psi(x) = -\frac{2mE}{\hbar^2}\psi(x)$$

Since particle is confined, $E < 0$

Let $k^2 = \frac{-2mE}{\hbar^2} > 0$

Then,

$$\boxed{\frac{d^2}{dx^2}\psi(x) = k^2\psi(x)} \quad (x \neq 0)$$

The solution is

$$\psi(x) = c_1 e^{kx} + c_2 e^{-kx}$$

Need to treat $x>0$ and $x<0$ separately as $V(x)$ discontinuous @ $x=0$.

We need $\int_{-\infty}^{\infty} dx\, \psi(x) = 1 \Rightarrow \lim_{x\to\pm\infty} \psi(x) = 0$

Therefore,

$$\boxed{\psi(x) = \begin{cases} c_1 e^{-kx} & x \geq 0 \\ c_2 e^{kx} & x \leq 0 \end{cases}}$$

From continuity of $\psi(x)$ @ $x=0$ we require:

$$c_1 = c_2$$

Normalization requires:

$$\int_{-\infty}^{\infty} dx \, |\psi(x)|^2 = 1$$

$$|a|^2 \left[\int_{-\infty}^{0} dx \, e^{2Kx} + \int_{0}^{\infty} dx \, e^{-2Kx} \right] = 1$$

$$|a|^2 \left[\frac{e^{2Kx}}{2K} \Big|_{-\infty}^{0} - \frac{e^{-2Kx}}{2K} \Big|_{0}^{\infty} \right] = 1$$

$$|a|^2 \left[\frac{1}{2K} - 0 - \left(0 - \frac{1}{2K} \right) \right] = 1$$

$$|a|^2 \frac{1}{K} = 1$$

$$c_1 = \sqrt{K}$$

Thus,

$$\boxed{\psi(x) = \begin{cases} \sqrt{K} \, e^{Kx} & x \leq 0 \\ \sqrt{K} \, e^{-Kx} & x \geq 0 \end{cases}}$$

We have
$$k^2 = \frac{-2mE}{\hbar^2}$$

$$\Rightarrow \boxed{E = \frac{-\hbar^2 k^2}{2m}}$$

Now, integrating Eq. (1) from $-\varepsilon$ to ε

$$-\frac{\hbar^2}{2m}\int_{-\varepsilon}^{\varepsilon} dx \frac{d^2\psi}{dx^2} - \beta \int_{-\infty}^{\infty} dx\, \delta(x)\psi(x) = \int_{-\varepsilon}^{\varepsilon} dx\, \psi(x)$$

$$-\frac{\hbar^2}{2m}\left(\frac{d\psi}{dx}\Big|_{\varepsilon} - \frac{d\psi}{dx}\Big|_{-\varepsilon}\right) - \beta\psi(0) = \int_{-\varepsilon}^{\varepsilon} dx\, \psi(x) \qquad (2)$$

Now,
$$\frac{d\psi}{dx} = \begin{cases} \sqrt{k^3}\, e^{kx} & x \le 0 \\ -\sqrt{k^3}\, e^{-kx} & x \ge 0 \end{cases}$$

Also, $\int_{-\varepsilon}^{\varepsilon} dx\, \psi(x) \xrightarrow{\varepsilon \to 0} 0$

$$\frac{d\psi}{dx}\Big|_{\varepsilon} - \frac{d\psi}{dx}\Big|_{-\varepsilon} \xrightarrow{\varepsilon \to 0} -2\sqrt{\frac{k^3}{2}}$$

Then, letting $\varepsilon \to 0$ in Eq. (2):

$$+\frac{\hbar^2}{\cancel{2}m}(\cancel{2}\sqrt{k^3} - \beta\sqrt{k} = 0$$

$$\frac{\hbar^2}{m}\sqrt{k^3} = \beta\sqrt{k}$$

$$\Rightarrow \beta = \frac{\hbar^2}{m}k$$

Then, $E = -\frac{\hbar^2}{2m}\left(\frac{m\beta}{\hbar^2}\right)^2$

$$E = \frac{-\cancel{\hbar^2}}{2\cancel{m}} \; \frac{\cancel{m^2}\beta^2}{\cancel{\hbar^4}} \quad 2$$

$$\boxed{E = -\frac{m\beta^2}{2\hbar^2}}$$

Alg

$$\boxed{\psi(x) = \begin{cases} \sqrt{\dfrac{m\beta}{\hbar^2}}\, e^{\frac{m\beta x}{\hbar^2}} & x \le 0 \\ \sqrt{\dfrac{m\beta}{\hbar^2}}\, e^{-\frac{m\beta x}{\hbar^2}} & x \ge 0 \end{cases}}$$