4-Vector

A 4-vector has four components, with indices $\mu = 0, 1, 2, 3$.

$$A^\mu = (A^0, A^1, A^2, A^3) = (A^0, \boldsymbol{A})$$

The zeroth component is called the 'time' component, component 1,2,3 the 'spatial' components. Latin indices like $i, j, k, etc.$ are usually reserved for the spatial components and can be $1, 2, 3$. The Greek indices like $\mu, \nu, \rho\ , etc.$ are from the both the time and spatial components and can be $0, 1, 2, 3$.

The 4-product is an inner product, but different from the usual Euclidean one. If A^μ and B^μ are 4-vectors their inner product:

$$\boxed{\begin{aligned} A \cdot B &= A^0B^0 - A^1B^1 - A^2B^2 - A^3B^3 \\ &= A^0B^0 - \boldsymbol{A} \cdot \boldsymbol{B} \\ &= A^0B^0 - |\boldsymbol{A}||\boldsymbol{B}| \cos\theta_{AB} \end{aligned}}$$

Sometimes the 4-product is also labeled as $A^\mu B_\mu$. Here θ_{AB} is the angle between the two 3-vectors $\boldsymbol{A}, \boldsymbol{B}$.

The 4-product is invariant. That is, it is the same for all inertial reference frames. It does not change under rotations or boosts. An example of a 4-vector is an infinitesimal displacement spacetime:

$$dr^\mu = (cdt, dx, dy, dz)$$

The c in the zeroth component makes all the components have units of length.

Lorentz Transformations

If we are in a reference frame S which sees object undergo a displace in spacetime of:

$$dr^\mu = (c\ dt, dx, dy, dz)$$

Another inertial reference frame, S', sees a displacement in spacetime of:

$$dr'^\mu = (c\ dt', dx', dy', dz')$$

If the S' is moving with respect to S with a speed of v in the positive x direction, then the two are related via:

Boost in x-direction

$$\begin{aligned} cdt' &= \gamma\left(cdt + \frac{v\ dx}{c^2}\right) \\ dx' &= \gamma(v\ dt + dx) \\ dy' &= dy \end{aligned}$$

4-Momentum

The 4-momentum is a 4-vector with:

$$p^{\mu} = (E, p^1, p^2, p^3)$$

The time component of p^{μ} is energy and the spatial components are the relativistic 3-momenta. For any scattering event we have two conditions:

1. *Conservation of 4-momentum*

$$\sum_{incoming} p^{\mu} = \sum_{outgoing} p^{\mu}$$

   This combines conservation of energy and conservation of 3-momenta.

2. *On mass-shell condition*

   For all incoming/outgoing particles we have:

$$p^2 = m^2c^2$$
$$\rightarrow E^2 - c^2|\boldsymbol{p}|^2 = m^2c^4$$

In the rest frame of a massive particle, $E = mc^2$.

$2 \rightarrow 2$ Scattering

In the special case of $AB \rightarrow CD$, this translates as:

1. $p_A^{\mu} + p_B^{\mu} = p_C^{\mu} + p_D^{\mu}$

Or, broken into time and spatial components:

$$E_A + E_B = E_C + E_D$$

$$\boldsymbol{p}_A + \boldsymbol{p}_B = \boldsymbol{p}_C + \boldsymbol{p}_D$$

2.

$$E_A^2 - c^2|\boldsymbol{p}_A|^2 = m_A^2c^4$$
$$E_B^2 - c^2|\boldsymbol{p}_B|^2 = m_B^2c^4$$
$$E_C^2 - c^2|\boldsymbol{p}_C|^2 = m_C^2c^4$$
$$E_D^2 - c^2|\boldsymbol{p}_D|^2 = m_D^2c^4$$