Classical Physics – Fall 2019

Section A: Mechanics

3) A particle of mass m moves in a central force with potential energy $U(r) = Kr^4$.

a. For the force to be attractive, what is the sign of K? Don't guess. Prove it.
b. What is the effective potential energy, $V(r)$?
c. For a given value of angular momentum, l, find r for a circular orbit.
d. What is the total energy of the circular orbit? Express your answer in terms of K and r_0 is the radius of the circular orbit.
e. What is the period of the circular orbit? Express your answer in terms of K, m, and r_0.

Solution)

a)

The central force is defined as $\boldsymbol{F} = -\nabla U$

$$\boldsymbol{F} = -\frac{\partial U}{\partial r}\hat{r} = -4Kr^3\hat{r}$$

This corresponds with a cubic restoring force. For positive K the minus sign ensures that the force is toward the center of force, for negative K the force will point away from the center of force. Thus:

$K > 0$: attractive

b)

The effective potential is defined as:

$$V(r) = U(r) + \frac{l^2}{2mr^2}$$

$$V(r) = Kr^4 + \frac{l^2}{2mr^2}$$

c)

The kinetic energy in polar coordinates is:

$$T = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

The Lagrangian has the form $L = T - U$:

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - Kr^4$$

The Lagrange equations of motion are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

For the generalized coordinate r:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - 4Kr^3$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \quad \rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r}$$

For the generalized coordinate θ:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \quad \rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta}$$

The equations of motion are:

$$m\ddot{r} - mr\dot{\theta}^2 + 4Kr^3 = 0 \qquad (1)$$

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0 \qquad (2)$$

For circular motion $r = r_0 = const$ $(\dot{r} = 0, \ddot{r} = 0)$ in (1):

$$-mr_0\dot{\theta}^2 + 4Kr_0^3 = 0 \qquad (3)$$

From (2):

$$mr^2\dot{\theta} = const = l$$

Plugin

$$\dot{\theta}^2 = \frac{l^2}{m^2r_0^4}$$

into (3):

$$-mr_0\frac{l^2}{m^2r_0^4} + 4Kr_0^3 = 0$$

$$4Kr_0^3 = \frac{l^2}{mr_0^3}$$

$$r_0^6 = \frac{l^2}{4mK}$$

d)

The total energy is:

$$E = T + U$$

$$E = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + Kr^4$$

At $r = r_0$:

$$E = \frac{m}{2}r_0^2\dot{\theta}^2 + Kr_0^4$$

where

$$\dot{\theta}^2 = \frac{l^2}{m^2 r_0^4} = \frac{4mKr_0^6}{m^2 r_0^4} = \frac{4Kr_0^2}{m}$$

$$E = \frac{m}{2} r_0^2 \frac{4Kr_0^2}{m} + Kr_0^4$$

$$E = 2Kr_0^4 + Kr_0^4$$

$$E = 3Kr_0^4$$

e)

Using the period of the orbits T:

$$\dot{\theta} = \frac{2\pi}{T}$$

where

$$\dot{\theta} = 2r_0 \sqrt{\frac{K}{m}}$$

We get:

$$T = \frac{2\pi}{\dot{\theta}}$$

$$T = \frac{\pi}{r_0} \sqrt{\frac{m}{K}}$$