Classical Physics – Fall 2018

Section B: Electricity and Magnetism

4) Consider the long cylindrical shell centered on the *z*-axis with inner radius a and outer radius b. The cylindrical shell caries a "frozen-in" magnetization, parallel to the *xy*-plane $\boldsymbol{M} = ks^2\hat{\phi}$ where k is a constant and s is the distance from the axis.

- **a.** Determine all bound current $\boldsymbol{j}_b$ and $\boldsymbol{K}_b$.
- **b.** Using the bound currents determined in part (a), determine the magnetic field $\boldsymbol{B}$ and the auxiliar field $\boldsymbol{H}$ for the regions $s < a$, $a < s < b$, and $s > b$.

Solution)

a)

Let's find the magnetization currents. The volume current is:

$$\boldsymbol{j}_b(\boldsymbol{r}) = \nabla \times \boldsymbol{M}(\boldsymbol{r})$$

Where the curl in cylindrical coordinate is:

$$\nabla \times \boldsymbol{M} = \left(\frac{1}{s}\frac{\partial M_z}{\partial \phi} - \frac{\partial M_\phi}{\partial z}\right)\hat{s} + \left(\frac{\partial M_s}{\partial z} - \frac{\partial M_z}{\partial s}\right)\hat{\phi} + \frac{1}{s}\left(\frac{\partial (sM_\phi)}{\partial s} - \frac{\partial M_s}{\partial \phi}\right)\hat{z}$$

Since $\boldsymbol{M} = ks^2\hat{\phi}$:

$$\nabla \times \boldsymbol{M} = \left(\frac{\partial M_\phi}{\partial z}\right)\hat{s} + \frac{1}{s}\left(\frac{\partial (sM_\phi)}{\partial s}\right)\hat{z} = \frac{1}{s}\left(\frac{\partial (ks^3)}{\partial s}\right)\hat{z} = 3ks\hat{z}$$

$$\boldsymbol{j}_b(\boldsymbol{r}) = 3ks\hat{z}$$

The surface currents are:

$$\boldsymbol{K}_b(\boldsymbol{r}_S) = \boldsymbol{M}(\boldsymbol{r}_S) \times \hat{n}(\boldsymbol{r}_S)$$

$$\boldsymbol{K}_b(\boldsymbol{r}_S = a) = \boldsymbol{M}(\boldsymbol{r}_S = a) \times \hat{n}(\boldsymbol{r}_S = a) = ka^2\hat{\phi} \times (-\hat{s}) = ka^2\hat{z}$$

$$\boldsymbol{K}_b(\boldsymbol{r}_S = b) = \boldsymbol{M}(\boldsymbol{r}_S = b) \times \hat{n}(\boldsymbol{r}_S = b) = kb^2\hat{\phi} \times (\hat{s}) = -kb^2\hat{z}$$

b)

The integral form of Ampere's law $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j}$ is:

$$\oint_C d\boldsymbol{l} \cdot \boldsymbol{B} = \mu_0 I_{enc}$$

The first integral is

$$\oint_C d\boldsymbol{l}\cdot\boldsymbol{B} = \oint_C B dl = \int_0^{2\pi} Bsd\phi = 2\pi s B$$

where $d\boldsymbol{l} = dl\hat{\phi} = sd\phi\hat{\phi}$.

Let's use cylindrical coordinates $x = s\cos\phi$, $x = s\sin\phi$, $z = z$. Then $d\boldsymbol{S} = sdsd\phi\hat{z}$:

- $s < a$

$$I_{enc} = 0$$

$$\boldsymbol{B}(s < a) = 0$$

- $a < s < b$

$$I_{enc} = \int_S d\boldsymbol{S}\cdot\boldsymbol{j}_b + \oint_{C:s=a} d\boldsymbol{l}\cdot\boldsymbol{K}_b(\boldsymbol{r}_S = a)\times\hat{n}(\boldsymbol{r}_S = a) = 3k\int_0^{2\pi} d\phi\int_a^s s'^2 ds'\hat{z}\cdot\hat{z} + ka^3\int_0^{2\pi} d\phi\,\hat{\phi}\cdot\hat{\phi}$$
$$= 2\pi k s^3 - 2\pi k a^3 + 2\pi k a^3 = 2\pi k s^3$$

Then:

$$2\pi s B = 2\pi\mu_0 k s^3 \quad \rightarrow \quad B = \mu_0 k s^2$$

- $s > b$

$$I_{enc} = \int_S d\boldsymbol{S}\cdot\boldsymbol{j}_b + \oint_{C:s=a} d\boldsymbol{l}\cdot\boldsymbol{K}_b(\boldsymbol{r}_S = a)\times\hat{n}(\boldsymbol{r}_S = a) + \oint_{C:s=b} d\boldsymbol{l}\cdot\boldsymbol{K}_b(\boldsymbol{r}_S = b)\times\hat{n}(\boldsymbol{r}_S = b) =$$
$$= 3k\int_0^{2\pi} d\phi\int_a^b s'^2 ds'\hat{z}\cdot\hat{z} + ka^3\int_0^{2\pi} d\phi\,\hat{\phi}\cdot\hat{\phi} - kb^3\int_0^{2\pi} d\phi\,\hat{\phi}\cdot\hat{\phi} =$$
$$= 2\pi k b^3 - 2\pi k a^3 + 2\pi k a^3 - 2\pi k b^3 = 0$$

Then:

$$2\pi s B = 0 \quad \rightarrow \quad B = 0$$

Thus:

$$\boldsymbol{B}(s) = \begin{cases} 0, & s < a \\ \mu_0 k s^2\hat{\phi}, & a < s < b \\ 0, & s > b \end{cases}$$

Using the fundamental relation of magnetic matter:

$$\boldsymbol{B}(\boldsymbol{r}) = \mu_0[\boldsymbol{M}(\boldsymbol{r}) + \boldsymbol{H}(\boldsymbol{r})]$$

$$\boldsymbol{H} = \frac{1}{\mu_0}\boldsymbol{B} - \boldsymbol{M}$$

$$\boldsymbol{H} = \begin{cases} 0, & s < a \\ 0, & a < s < b \\ 0, & s > b \end{cases}$$