Equations for Physics Ph.D. Qualifying Exam

Note: Not exhaustive.

Classical Mechanics

Lagrangian Mechanics

- Lagrangian: $L = T - V$
- Euler-Lagrange equations: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$ Conserved quantities: $\frac{\partial L}{\partial q_i} = 0 \rightarrow \ p_i = \frac{\partial L}{\partial \dot{q}_i} = const.$
- Kinetic energy for a single particle:

$$
\begin{aligned}
T &= \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) && (Cartesian) \\
&= \frac{1}{2} m\left(\dot{r}^2 + r^2\,\dot{\theta}^2 + \dot{z}^2\right) && (Cylindrical) \\
&= \frac{1}{2} m\left(\dot{r}^2 + r^2\,\dot{\theta}^2 + r^2 \sin^2\theta\ \dot{\phi}^2\right) && (Spherical)
\end{aligned}
$$

- If $\frac{\partial L}{\partial t} = 0 \rightarrow E = T + V = const.$

Rotational Dynamics

- Moment of Inertia: $I = \int dm\, r^2$ Parallel axis theorem: $I = I_{CM} + M\ell^2$
- Kinetic Energy: $T = \frac{1}{2}\, I\, \dot{\theta}^2$
- Torque: $\boldsymbol{\tau} = I\boldsymbol{\alpha} = \boldsymbol{F} \times \boldsymbol{r}$

Harmonic Motion and Small oscillations

- 1D equation for small oscillations: $\frac{d\theta}{dt} \approx -\omega^2\theta \rightarrow \quad \theta(t) = c_1 \cos\omega t + c_2 \sin\omega t$
- General: For $M_{ij} = \frac{\partial^2 T}{\partial x_i\, \partial x_j}$ and $V_{ij} = \frac{\partial^2 V}{\partial x_i\, \partial x_j}$ (evaluated at equilibrium point and zero velocity), frequencies found with: $\det(V - \omega^2 M) = 0$

Central Force

For $V(\boldsymbol{r}) = V(|\boldsymbol{r}|) = V(r)$

- Angular momentum conserved: $l = mr^2\,\dot{\theta}$ (From E-L equations)
- Equation of motion: $m\,\ddot{r} = mr\,\dot{\theta}^2 - \frac{dV(r)}{dr} = \frac{l^2}{mr^3} - \frac{dV(r)}{dr} \quad \rightarrow \ U_{eff}(r) = \frac{l^2}{2mr^2} + V(r)$
- First order equation of motion: use $E = const.$, $u = 1/r$ and $\frac{d}{dt} = \frac{d\theta}{dt}\frac{d}{d\theta}$

Miscellaneous

- $W = \int \boldsymbol{F} \cdot d\boldsymbol{l} = \Delta KE$

Electromagnetism

Maxwell's Equations and Related

	Differential form	Integral form
Gauss' Law	$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}$	$\int \boldsymbol{E} \cdot d\boldsymbol{A} = \frac{Q_{enc}}{\epsilon_0}$
Gauss' Law for Magnetism	$\nabla \cdot \boldsymbol{B} = 0$	$\int \boldsymbol{B} \cdot d\boldsymbol{A} = 0$
Faraday's Law	$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$	$\oint_{\partial\Sigma} \boldsymbol{E} \cdot d\boldsymbol{\ell} = -\frac{d}{dt}\int_{\Sigma} \boldsymbol{B} \cdot d\boldsymbol{A}$
Ampere's Law (+ Maxwell correction)	$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_o \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$	$\oint_{\partial\Sigma} \boldsymbol{B} \cdot d\boldsymbol{\ell} = \mu_0 I_{enc} + \mu_o \epsilon_0 \frac{d}{dt}\int_{\Sigma} \boldsymbol{E} \cdot d\boldsymbol{A}$

- Potential: $V(r) = \frac{1}{4\pi\epsilon_0} \int_{Vol} d^3\boldsymbol{r}' \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}\prime|}$ $\quad A(r) = \frac{\mu_0}{4\pi} \int_{Vol} d^3\boldsymbol{r}' \frac{\boldsymbol{J}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}\prime|}$
- Biot-Savart Law: $B(r) = \frac{\mu_0}{4\pi} \int_C \frac{I\, d\boldsymbol{\ell} \times (\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}$
- Lorentz Force: $\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$ Poynting Vector: $\boldsymbol{S} = \frac{\boldsymbol{E}\times\boldsymbol{B}}{\mu_0}$

Electrostatics and Potential

- Conductors: $\boldsymbol{E}_{inside} = \boldsymbol{0} \rightarrow \rho_{inside} = 0$, charge on surface, conductor is an equipotential, $\boldsymbol{E}_{just\ outside} \perp$ surface and $\sigma = -\epsilon_0 \frac{\partial V}{\partial n}$
- Laplace's equation: $\nabla^2 V(\boldsymbol{r}) = 0$
- Solution of Laplace's Equation in spherical co-ordinates with azimuthal symmetry:

$$V(r, \cos\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

- Electric dipole: $\boldsymbol{p}(\boldsymbol{r}) = \int_{Vol} d^3\boldsymbol{r}' \; \rho(\boldsymbol{r}')(\boldsymbol{r}' - \boldsymbol{r})$ $(contin.)$ $= \sum_i q_i\, (\boldsymbol{r_i} - \boldsymbol{r})$ $(discrete)$
- Electric quadrupole: $Q_{ij}(\boldsymbol{r}) = \int_{Vol} d^3\boldsymbol{r}' \; \rho(\boldsymbol{r}')\left(3r_i r_j - |\boldsymbol{r}|^2]\, \delta_{ij}\right)$ $(contin.)$ $=$ $\sum_i q_i \left(3r_i r_j - |\boldsymbol{r}|^2]\, \delta_{ij}\right)$ $(discrete)$
- Magnetic dipole: $\boldsymbol{m} = I \int d\boldsymbol{a}$

Capacitance and Electric fields in matter

- Capacitance $C = dQ/dV$
- Displacement: $\boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P}$ $\quad \nabla \cdot \boldsymbol{D}(\boldsymbol{r}) = \rho_{free}(\boldsymbol{r})$
- Linear dielectric: $\boldsymbol{P} = \epsilon_0\, \chi\, \boldsymbol{E}$ $\quad \boldsymbol{D} = \epsilon \boldsymbol{E}$

Magnetic fields in matter

- Volume Current: $\boldsymbol{j_b}(\boldsymbol{r}) = \nabla \times \boldsymbol{M}(\boldsymbol{r})$ Surface current: $\boldsymbol{K_b}(\boldsymbol{r}) = \boldsymbol{M}(\boldsymbol{r}) \times \hat{\boldsymbol{n}}(\boldsymbol{r})$
- Magnetic field: $\boldsymbol{B}(\boldsymbol{r}) = \mu_0[\boldsymbol{M}(\boldsymbol{r}) + \boldsymbol{H}(\boldsymbol{r})]$

Statistical Mechanics and Thermodynamics

Statistical Mechanics

- Boltzmann Entropy: $S = k_B \ln \Omega$
- Stirling's Approximation: $\ln N! \approx N \ln N - N$ $\quad (N \gg 1)$
- Partition function $Z = \sum_i e^{-E_i \beta}$ $\quad$ Boltzmann probability: $p_i = \frac{e^{-E_i\beta}}{Z}$ $\quad$ $(\beta = \frac{1}{k_B T})$
- System with N identical subcomponents $Z = \prod_{i=1}^{N} z_i = z^N$
- Energy: $U = -\frac{\partial \ln Z}{\partial \beta}$ $\quad$ Free energy: $F = U - TS = -k_B T \ln Z$
- Heat capacity: $C_V = \left(\frac{\partial Q}{\partial T}\right)_V$
- Fermi statistics: $\mathcal{Z} = \sum_{n=0,1} e^{-(\epsilon-\mu)n\beta} = 1 + e^{-(\epsilon-\mu)\beta}$ $\quad$ $\bar{n} = \sum_{n=0,1} nP(n) = \frac{1}{e^{(\epsilon-\mu)\beta}+1}$
- Bose statistics: $\mathcal{Z} = \sum_{n=0}^{\infty} e^{-(\epsilon-\mu)n\beta} = \frac{1}{1-e^{-(\epsilon-\mu)\beta}}$ $\quad$ $\bar{n} = \sum_{n=0}^{\infty} nP(n) = \frac{1}{e^{(\epsilon-\mu)\beta}-1}$

Thermodynamics

- First Law: $dU(S,V) = \delta Q + dW = TdS - PdV$
- Enthalpy $H = H(S,P) = U + PV$ $\quad$ Helmholtz Free Energy: $F = F(T,V) = U - TS$
- Gibbs Free Energy: $G = G(T,P) = H - TS = U + PV - TS$
- Maxwell relation for U: $T = \left(\frac{\partial U}{\partial S}\right)_V$ & $P = -\left(\frac{\partial U}{\partial V}\right)_S \rightarrow \frac{\partial^2 U}{\partial S \partial V} = \left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial P}{\partial S}\right)_V$
- Can get other Maxwell relations from $H, F,$ & G

- Adiabatic: $\delta Q = 0$ $\quad$ Isochoric: $dV = 0$ $\quad$ Isobaric: $dP = 0$
- Ideal gas: $PV = nRT = N\, k_B T,\quad U = C_V nRT,\quad PV^{\gamma} = const.$ $\quad$ $(adiabtic\ process,\ \gamma = \frac{C_P}{C_V})$
- Equipartition theorem: $U = N\ f\frac{1}{2}k_B T$ $\quad$ $(f = \#\ of\ degrees\ of\ freedom)$
- Incompressible Fluid: $dV = 0$

Quantum Mechanics

Expectation & Eigenvalues:

- Expectation value of operator: $\langle \hat{A} \rangle = \langle \psi | \hat{A} |\, \psi \rangle$
- Eigenvalue equation: $\hat{A}|\psi\rangle = \lambda\, |\psi\rangle$

Time-independent Schrödinger Equation

$$-\frac{\hbar^2}{2m}\nabla^2\,\psi(\boldsymbol{r}) + V(\boldsymbol{r})\psi(r) = E\psi(\boldsymbol{r}) \tag{1}$$

Normalization: $\langle\psi|\psi\rangle = \int d^n\boldsymbol{r}\,\psi^*(\boldsymbol{r})\psi(\boldsymbol{r}) = 1$

Time-dependent Schrödinger Equation

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}(t)\;|\psi(t)\,\rangle$$

- Time dependent solution to Eq. (1): $\psi(\boldsymbol{r},t) = \sum_n e^{-iE_nt/\hbar}\,c_n\,\psi_n(\boldsymbol{r})$

Exact Solutions:

- Particle in Box: Solve Eq. (1), sine solutions, impose boundary conditions. $E_n \sim n^2$
- Hamonic Oscillator: use annihilation and creation operator. Energy: $E_n = \hbar\omega(n+\frac{1}{2})$
- Hydrogen Atom $E_n = -Z^2\frac{13.6\ eV}{n^2}$ States: $|n\;l\;m\,\rangle$
- Spin-1/2 particle in a magnetic field:

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = -\boldsymbol{\mu}\cdot\boldsymbol{B}\,|\psi(t)\,\rangle = -\frac{gq}{2m}\,\frac{\hbar}{2}\boldsymbol{\sigma}\cdot\boldsymbol{B}$$

For electron $g = 2$ (approx.) and $q = -e$:

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \frac{\hbar\omega}{2}\;\boldsymbol{\sigma}\cdot\boldsymbol{B}\;|\psi(t)\,\rangle$$

where $\omega = \frac{e|\boldsymbol{B}|}{m}$ (note: $\boldsymbol{B} = |\boldsymbol{B}|\,\hat{B}$).

Unbound states and Scattering

- Transmission coefficient: $T = \frac{|F|^2}{|A|^2}$ Refection coefficient: $T = \frac{|B|^2}{|A|^2}$
- $R + T = 1$
- $e^{ikx} \rightarrow +\,moving\;\;plane\;wave$ & $e^{-ikx} \rightarrow -\,moving\;\;plane\;wave$
  (Plug into $\psi(\boldsymbol{r},t) = \sum_n e^{-iE_nt/\hbar}\,c_n\,\psi_n(\boldsymbol{r})$ to see)

Identical Particles

- Bosons: symmetric states $P\,|12\rangle = |21\rangle$
- Fermions: antisymmetric states $P\,|12\rangle = -|21\rangle$
- 2 particles position wavefunction: $\psi^{S/A}_{space}(x_1,x_2) = N^{S/A}\,[\phi_a(x_1)\phi_b(x_2) \pm \phi_b(x_1)\phi_a(x_2)]$

Time-independent Perturbation Theory

For Exact + Perturbation Hamiltonian: $H = H_0 + H'$

- First order corrections: $E_n^{(1)} = \langle n^{(0)} | H' | n^{(0)} \rangle$ $\quad |\psi_n^{(1)}\rangle = \sum_{m\neq n} \frac{\langle n^{(0)}|H'|m^{(0)}\rangle}{E_n^{(0)} - E_m^{(0)}} |\psi_n^{(0)}\rangle$
- Second order corrections: $E_n^{(2)} = \sum_{m\neq n} \frac{|\langle n^{(0)}|H'|m^{(0)}\rangle|^2}{E_n^{(0)} - E_m^{(0)}}$

Time Dependent Perturbation Theory

$$c_k(t) = \frac{1}{i\hbar}\int_0^t \langle k|H'(t')|i\rangle e^{i\,(E_k - E_i)t/\hbar} dt'$$

Modern Physics

Special relativity

- Lorentz transformation (x-direction, $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$):

$$dt' = \gamma\left(cdt - \frac{vdx}{c^2}\right)$$
$$dx' = \gamma(dx - vdt)$$
$$dy' = dy$$
$$dz' = dz$$

- Time dilation: $t' = \gamma t$ $\quad$ Length contraction: $L' = \frac{x}{\gamma} = L\sqrt{1 - v^2/c^2}$
- Relativistic velocity addition (can get using Lorentz transformation):

$$u'_x = \frac{dx'}{dt'} = \frac{\frac{dx}{dt} - v}{1 - \frac{v\, dx/dt}{c^2}} = \frac{u_x - v}{1 - \frac{vu_x}{c^2}}$$

- Doppler Shift (can get with Lorentz, on-shell and Planck relation):

$$f' = \sqrt{\frac{v+c}{v-c}} f$$

- 4-product: $A \cdot B = A^0\, B^0 - \ \boldsymbol{A} \cdot \boldsymbol{B} = \ A^0\, B^0 - \ |\boldsymbol{A}||\boldsymbol{B}| \cos\theta_{AB}$

Relativistic Elastic Scattering

- 4-momemtum: $p^\mu = \left(\frac{E}{c},\ p^x,\ p^y, p^z\right)$
- Energy/momentum conservation: $\quad \sum_{in} p^\mu = \sum_{out} p^\mu$
- On-mass shell condition (dispersion relation): $\quad c^2p^2 = E^2 - c^2|\boldsymbol{p}|^2 = m^2c^4$
- Energy/momentum: $E = \gamma mc^2$ $\quad \boldsymbol{p} = \gamma m \boldsymbol{v}$ $\quad \frac{|\boldsymbol{p}|}{E} = \frac{|\boldsymbol{v}|}{c^2}$
- Kinetic Energy: $KE = E - \ mc^2$

Miscellaneous

- Compton scattering: $\lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\theta)$

  To derive use $\left(p_\gamma^\mu - p_\gamma'^\mu\right)^2 = \left(-p_e^\mu + p_e'^\mu\right)^2$, conservation of energy and Planck's relation
- Bohr's quantization condition: $l = pr = mvr = n\hbar$

  Plug into F for circular orbit and Coloumb attraction. Use to get $E \sim n^2$
- Planck relation: $E = hf = \hbar\omega$

Uncertainties

- Uncorrelated: $\Delta f = \sqrt{\left(\frac{\partial f}{\partial x_1}\right)^2 (\Delta x_1)^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 (\Delta x_2)^2 + \cdots \left(\frac{\partial f}{\partial x_n}\right)^2 (\Delta x_n)^2}$

Mathematical Relations

- Binomial expansion: $(1+\epsilon)^n = 1 + n\epsilon + \mathcal{O}(\epsilon^2)$ $(|\epsilon| \ll 1)$
- Stirling's approximation: $\ln N! \approx N \ln N - N$ $(N \gg 1)$
- Pauli Matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Variance: $\langle (x-\mu)^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2$
- Volume element: $dV = dx\, dy\, d = r\, dr\, d\theta\ (Cylrdrical) = r^2\, d\cos\theta\ d\phi\ (Spherical)$

- Fourier relations:

$$\int_{-L}^{L} dx\ \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right) = L\delta_{mn} = \int_{-L}^{L} dx\ \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

$$\int_{-L}^{L} dx\ \cos\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right) = 0$$

- Legendre polynomial orthogonality:

$$\int_{-1}^{1} dx\, P_l(x) P_{l'}(x) = \frac{2}{2l+1}\delta_{l,l'}$$

- Divergence theorem: $\int \nabla \cdot \boldsymbol{A}\, dV = \oint \boldsymbol{A} \cdot d\boldsymbol{a}$
- Curl theorem: $\int (\nabla \times \boldsymbol{A}) \cdot d\boldsymbol{a} = \oint \boldsymbol{A} \cdot d\boldsymbol{l}$
- $\nabla \cdot (\nabla \times \boldsymbol{A}) = 0$