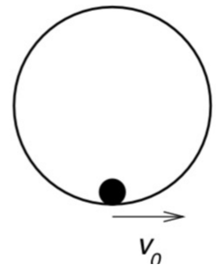
2007 Fall - A3

Problem A3: A particle of mass m is moving without friction inside of a vertical circular track of radius R. When it is at its lowest position, its speed is v_0. Use the Lagrange method to find the minimum value of v_0 for which the particle will go completely around the circle without losing contact with the track?



$$L = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) - mgr\sin\theta + \lambda (r - a)$$

Took center of circle as origin.

$\underline{r}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r}$$

$$\frac{d}{dt}\left(m\dot{r} \right) = \frac{1}{2} m (2r)\dot{\theta}^2 - mg\sin\theta + \lambda$$

$$m\ddot{r} = mr\dot{\theta}^2 - mg\sin\theta + \lambda \quad (1)$$

$\underline{\lambda}$

$$r - a = 0 \Rightarrow \ddot{r} = 0$$

Plugging these two results into Eq.(1),

$$0 = ma\dot{\theta}^2 - mg\sin\theta + \lambda$$

$$\lambda = mg\sin\theta - ma\dot{\theta}^2 \quad (2)$$

Now, since $\frac{\partial L}{\partial t} = 0$, $E = T + V$ is conserved.

Initially,

$$E = \frac{1}{2} m v_0^2 - mga$$

Later,

$$\frac{1}{2} m v_0^2 - mga = \frac{1}{2} m a^2 \dot{\theta}^2 + mg a \sin\theta$$

$$\Rightarrow \quad ma\dot{\theta}^2 = \frac{m v_0^2}{a} - mg - 2mg \sin\theta \quad (3)$$

Inserting Eq. (3) into Eq. (2),

$$\lambda = mg \sin\theta - \left(\frac{m v_0^2}{a} - 2mg - 2mg \sin\theta\right)$$

$$\lambda = 3mg \sin\theta + 2mg - \frac{m v_0^2}{a}$$

$$\lambda = mg(3\sin\theta + 2) - \frac{m v_0^2}{a}$$

Now, λ is the normal force. We need

$$\lambda \geq 0$$

$$\cancel{m}g(3\sin\theta+2) - \frac{\cancel{m}v_0^2}{a} \geq 0$$

$$g(3\sin\theta+2) \geq \frac{v_0^2}{a}$$

$$ga(3\sin\theta+2) \geq v_0^2 \qquad (4)$$

Eq. (4) must hold for all $\sin\theta$.
Since the max of $\sin\theta$ is 1 then,

$$5ga \geq v_0^2$$

$$\Rightarrow \boxed{V_{0,min} = \sqrt{5ga}}$$