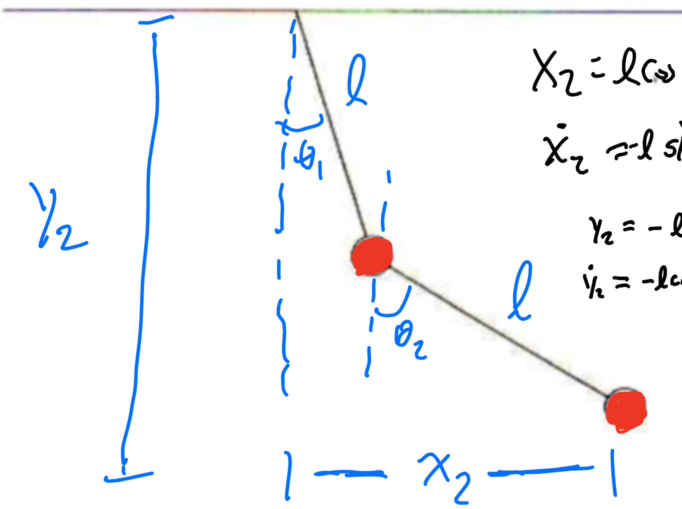
CM2: Two identical simple pendulums (length l and mass m) are coupled together as shown in the figure. Consider small oscillations, calculate the characteristic frequencies and derive the normal modes.



$$X_2 = l\cos\theta_1 + l\cos\theta_2$$

$$\dot{X}_2 = -l\sin\theta_1\dot{\theta}_1 - l\sin\theta_2\,\dot{\theta}_2$$

$$Y_2 = -l\sin\theta_1 - l\sin\theta_2$$

$$\dot{Y}_2 = -l\cos\theta_1\,\dot{\theta}_1 - l\cos\theta_2\,\dot{\theta}_2$$

Now

$$\dot{X}_2^2 + \dot{Y}_2^2$$

$$= (-l\cos\theta_1\,\dot{\theta}_1 - l\cos\theta_2\,\dot{\theta}_2)^2 + (-l\sin\theta_1\,\dot{\theta}_1 - l\sin\theta_2\,\dot{\theta}_2)^2$$

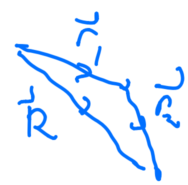
$$= l^2\cos^2\theta_1\,\dot{\theta}_1^2 + 2l^2\cos\theta_1\cos\theta_2\,\dot{\theta}_1\dot{\theta}_2 + l^2\cos^2\theta_2\,\dot{\theta}_2^2 + l^2\sin^2\theta_1\,\dot{\theta}_1^2 + 2l^2\sin\theta_1\sin\theta_2\,\dot{\theta}_1\dot{\theta}_2 + l^2\sin^2\theta_2\,\dot{\theta}_2^2$$

$$= l^2\dot{\theta}_1^2 + l^2\dot{\theta}_2^2 + 2l^2\dot{\theta}_1\dot{\theta}_2\,(\underbrace{\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2}_{\cos(\theta_1-\theta_2)})$$

$$= l^2\dot{\theta}_1^2 + l^2\dot{\theta}_2^2 + 2l^2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)$$

Note: Could have also obtained via:

$$|\vec{R}|^2 = |\vec{r}_1|^2 + |\vec{r}_2|^2 - 2|\vec{r}_1||\vec{r}_2|\cos\theta_{12}$$



Then,

$$T = \frac{1}{2} m \ell^2 \dot{\theta}_1^2 + \frac{1}{2} m (\dot{x}_2^2 + \dot{y}^2)$$

$$\boxed{T = \frac{1}{2} m\ell^2 \left[2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right]} \quad (1)$$

Also

$$V = mg y_1 + mg y_2$$

$$V = -mg\ell\cos\theta_1 - mg\ell\cos\theta_1 - mg\ell\cos\theta_2$$

$$\boxed{V = -2mg\ell\cos\theta_1 - mg\ell\cos\theta_2} \quad (2)$$

Calculate mass matrix, M:

$$M = \begin{bmatrix} \frac{\partial^2 T}{\partial \dot{\theta}_1^2} & \frac{\partial^2 T}{\partial \dot{\theta}_1 \dot{\theta}_2} \\ \frac{\partial^2 T}{\partial \dot{\theta}_2 \partial \dot{\theta}_1} & \frac{\partial^2 T}{\partial \dot{\theta}_2^2} \end{bmatrix} \Bigg|_{\substack{\theta_1, \theta_2 = 0 \\ \dot{\theta}_1, \dot{\theta}_2 = 0}}$$

$$M = \begin{bmatrix} 2ml^2 & ml^2 \\ ml^2 & ml^2 \end{bmatrix} \Bigg|_{\substack{\theta_1, \theta_2 = 0 \\ \dot{\theta}_1, \dot{\theta}_2 = 0}}$$

The stiffness matrix

$$U = \begin{bmatrix} 2mgl & 0 \\ 0 & mgl \end{bmatrix} \Bigg|_{\substack{\theta_1, \theta_2 = 0 \\ \dot{\theta}_1, \dot{\theta}_2 = 0}}$$

$$\det(-\omega^2 M + U) = 0$$

$$\begin{vmatrix} -2\cancel{m}l^{\cancel{2}}\omega^2 + 2\cancel{m}g\cancel{l} & -\cancel{m}l^{\cancel{2}}\omega^2 \\ -\cancel{m}l^{\cancel{2}}\omega^2 & -\cancel{m}l^{\cancel{2}}\omega^2 + \cancel{m}g\cancel{l} \end{vmatrix} = 0$$

$$\begin{vmatrix} -2l\omega^2 + 2g & -l\omega^2 \\ -l\omega^2 & -l\omega^2 + g \end{vmatrix} = 0$$

$$(-2l\omega^2 + 2g)(-l\omega^2 + g) - l^2\omega^4 = 0$$

$$2l^2\omega^4 - 2l\omega^2 g - 2l\omega^2 g + 2g^2 - l^2\omega^4 = 0$$

$$l^2\omega^4 - 4lg\omega^2 + 2g^2 = 0$$

$$\omega^4 - \frac{4g}{l}\omega^2 + 2\frac{g^2}{l^2} = 0$$

$$\omega^2 = \frac{\frac{4g}{l} \pm \sqrt{16g^2/l^2 - 4\left(\frac{2g^2}{l^2}\right)}}{2}$$

$$\omega^2 = \frac{2g}{l} \pm \frac{1}{2}\sqrt{8\frac{g^2}{l^2}}$$

$$\omega^2 = \frac{2g}{l} \pm \frac{1}{2}\left(\frac{2g}{l}\right)\sqrt{2}$$

$$\boxed{\omega_{1,2}^2 = \left(2 \pm \sqrt{2}\right)\frac{g}{l}}$$

Normal modes satisfy:

$$(-\omega^2 M + U)\vec{v} = 0$$

Let $\frac{g}{l} = \omega_0$

$$\left[-\omega^2 \begin{pmatrix} 2ml^2 & ml^2 \\ ml^2 & ml^2 \end{pmatrix} + \begin{pmatrix} 2mgl & 0 \\ 0 & mgl \end{pmatrix}\right]\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2ml^2\omega^2 + 2mgl & -ml^2\omega^2 \\ -ml^2\omega^2 & -ml^2\omega^2 + mgl \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Divide by ml^2

$$\begin{pmatrix} -2\omega^2 + 2g/l & -\omega^2 \\ -\omega^2 & -\omega + g/l \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2\omega^2+2\omega_0^2 & -\omega^2 \\ -\omega^2 & -\omega^2+\omega_0^2 \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Just need top row

$$(-2\omega^2+2\omega_0)V_1 - \omega^2 V_2 = 0$$

$$\frac{V_2}{V_1} = \frac{2(-\omega^2+\omega_0^2)}{\omega^2}$$

ω_1

$$\frac{V_2}{V_1} = \frac{2(-2-\sqrt{2}+1)\cancel{\omega_0^2}}{(2+\sqrt{2})\cancel{\omega_0^2}}$$

$$= \frac{-2-2\sqrt{2}}{2+\sqrt{2}}\;\frac{2-\sqrt{2}}{2-\sqrt{2}}$$

$$= \frac{-4+2\sqrt{2}-4\sqrt{2}+4}{4-2}$$

$$= \frac{-2\sqrt{2}}{2}$$

$$= -\sqrt{2}$$

$$\Rightarrow \boxed{\begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix} = A_1\begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}\cos(\omega_1 t+\delta)}$$

ω_2

$$\frac{v_2}{v_1} = \frac{2\left(-2+\sqrt{2}+1\right)\omega_0^2}{\left(2-\sqrt{2}\right)}$$

$$= \frac{-2+2\sqrt{2}}{2-\sqrt{2}} \quad \frac{2+\sqrt{2}}{2+\sqrt{2}}$$

$$= \frac{-4 -2\sqrt{2}+4\sqrt{2}+4}{4-2}$$

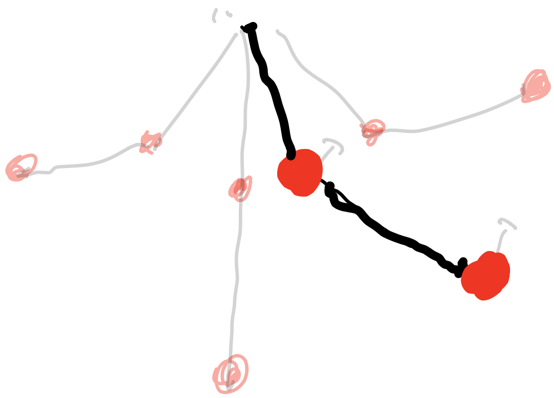
$$= \frac{2\sqrt{2}}{2}$$

$$= \sqrt{2}$$

$$\boxed{\begin{pmatrix}\theta_1(t)\\ \theta_2(t)\end{pmatrix} = A_2\begin{pmatrix}1\\ \sqrt{2}\end{pmatrix}\cos\left(\omega_2 t+\delta\right)}$$

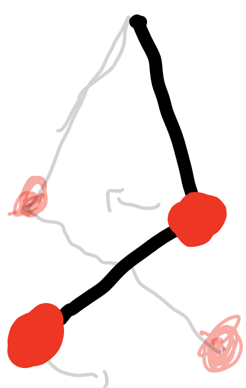
We can picture the two normal modes:

ω_2



The two masses move together, with th second's angle ~40% larger ($\sqrt{2}:1$)

ω_1



Masses move in opposite directions. Magnitude, again, is ~40% larger for second one.

Faster than above mode

All motion of small oscillations are liner combo of these two normal modes. Qualitatively similar case is gif:

https://personal.math.ubc.ca/~israel/m215/doublpen/doublpen.html