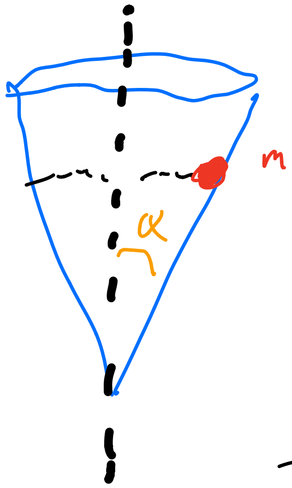
Classical 2009 - A4

Problem CM3: A particle of mass m moves on the surface of a circular cone of half angle α with its axis in the vertical direction. Gravity acts downward.

a) Write the equations of motion and find the conditions for motion of the particle to remain at a constant height h above the cone's vertex.

b) Find the frequency of small oscillations about this horizontal trajectory.



$$\tan\alpha = \frac{r}{z}$$

$$r = \tan\alpha \, z \qquad (1)$$

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) \qquad (2)$$

Insert (1) into (2)

$$T = \frac{1}{2}m\left(\tan^2\alpha\,\dot{z}^2 + z^2\tan^2\alpha\,\dot{\theta}^2 + \dot{z}^2\right)$$

Note: $1 + \tan^2\alpha = \frac{1}{\cos^2\alpha}$

$$T = \frac{1}{2}m\left(\frac{\dot{z}^2}{\cos^2\alpha} + \tan^2\alpha\, z^2\dot{\theta}^2\right)$$

Also,

$$V = mgz$$

$$\Rightarrow \quad L = T - V = \frac{1}{2}m\left(\frac{\dot{z}^2}{\cos^2\alpha} + \tan^2\alpha\, z^2 \dot{\theta}^2\right) - mgz$$

$\underline{\theta}$ Since $\frac{\partial L}{\partial \theta} = 0$ we have a constant of motion:

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}m \tan^2\alpha\, z^2 (2\dot{\theta})$$

$$\boxed{p_\theta = m \tan^2\alpha\, z^2 \dot{\theta}} \qquad (3)$$

$\underline{z}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = \frac{\partial L}{\partial z}$$

$$\frac{d}{dt}\left(\frac{\cancel{m}\dot{z}}{\cos^2\alpha}\right) = \cancel{m}\tan^2\alpha\, z\,\dot{\theta}^2 - \cancel{m}g$$

$$\frac{\ddot{z}}{\cos^2\alpha} = \tan^2\alpha\, z\,\dot{\theta}^2 - g$$

$$\boxed{\ddot{z} = \sin^2\alpha \, z \, \dot{\theta}^2 - g\cos^2\alpha} \quad (4)$$

Note: this can be simplified further by using Eq(3) to get rid of $\dot{\theta}^2$ in Eq (4), but it asks for just the equations (plural).

At $z = h = \text{const.} \Rightarrow \dot{z} = 0, \ \ddot{z} = 0$

$$0 = \sin^2\alpha \, h \, \dot{\theta}_c^2 - g\cos^2\alpha$$

$$g\cos^2\alpha = \sin^2\alpha \, h \, \dot{\theta}_c^2$$

$$\Rightarrow \quad \dot{\theta}_c^2 = \frac{g}{h}\frac{1}{\tan^2\alpha}$$

Critical θ for horizontal motion

$$\boxed{\dot{\theta}_c = \pm \frac{1}{\tan\alpha}\sqrt{\frac{g}{h}}} \quad (5)$$

b) From (3)

$$\dot{\theta}^2 = \frac{p_\theta^2}{m^2\tan^4\alpha \, z^4}$$

Then,

$$\ddot{z} = \sin^2\alpha \frac{p_\theta^2}{m^2\tan^4\alpha}\frac{1}{z^3} - g\cos^2\alpha$$

$$\ddot{z} = \frac{\cos^4\alpha \, p_\theta^2}{m^2 \sin^2\alpha} \frac{1}{z^3} - g\cos\alpha$$

Let $z = h + x \qquad |x| \ll h$

$$\Rightarrow \quad \ddot{x} = \frac{\cos^4\alpha \, p_\theta^2}{m^2 \sin^2\alpha} \frac{1}{(h+x)^3} - g\cos\alpha$$

Now

$$\frac{1}{(h+x)^3} = \frac{1}{h^3} \frac{1}{(1+x/h)^3}$$

$$\approx \frac{1}{h^3}\left(1 - \frac{3x}{h}\right)$$

$$= \frac{1}{h^3} - \frac{3x}{h^4}$$

Then,

$$\ddot{x} = \frac{\cos^4\alpha \, p_\theta^2}{m^2 \sin^2\alpha}\left(\frac{1}{h^3} - \frac{x}{h^4}\right) - g\cos^2\alpha \quad (6)$$

Now

$$P_\theta = m \tan^2\alpha \, h^2 \dot{\theta}_c \qquad (7)$$

Insert (5) into (7),

$$P_\theta = m \tan^2\alpha \, h^2 \left(\pm \frac{1}{\tan\alpha} \sqrt{\frac{g}{h}} \right)$$

$$P_\theta = \pm m \tan\alpha \sqrt{g h^3}$$

$$P_\theta^2 = m^2 \tan^2\alpha \, g h^3 \qquad (8)$$

Insert Eq.(8) into (6),

$$\ddot{x} = \frac{\cos^4\alpha}{\cancel{m^2} \sin^2\alpha} \left(\cancel{m^2} \tan^2\alpha \, g h^3 \right) \left(\frac{1}{h^3} - \frac{3x}{h^4} \right) - g\cos^2\alpha$$

$$\ddot{x} = \cos^2\alpha \, g h^3 \left(\frac{1}{h^3} - \frac{3x}{h^4} \right) - g\cos^2\alpha$$

$$= \cos^2\alpha \, g - 3 \frac{g\cos^2\alpha}{h} x - g\cos^2\alpha$$

$$\ddot{x} = \frac{-3g\cos^2\alpha}{h} x$$

The solution is sinusoidal with

$$\omega^2 = \frac{3g\cos^2\alpha}{h}$$

$$\Rightarrow \boxed{\omega = \cos\alpha \sqrt{\frac{3g}{h}}}$$