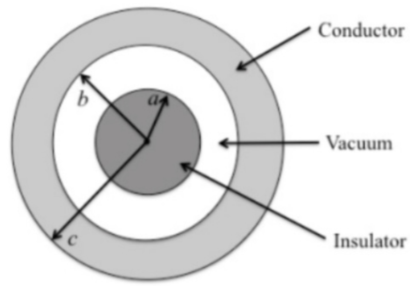
EM2: An insulating sphere of radius a carries a charge density of

$$\rho = \frac{k}{r}$$

in the region $r \leq a$, where k is a positive constant and r is the distance from the center of the sphere. The sphere is surrounded by a thick, concentric conducting metal shell with an inner radius b and an outer radius c. The conducting shell carries no net charge.



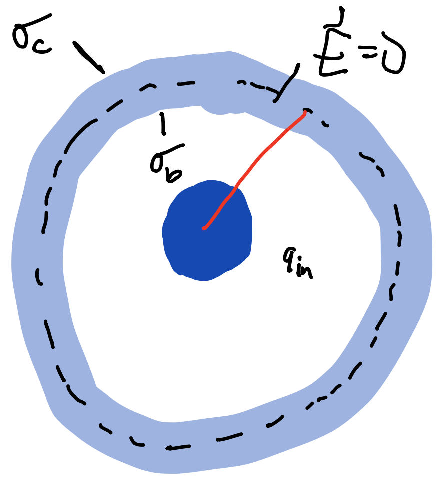
a. Determine the surface charge density at the inner ($r = b$) and outer surfaces ($r = c$) of the conductor.

b. Determine the electric field $\vec{E}$ in all four regions (i) $r < a$, (ii) $a < r < b$, (iii) $b < r < c$, and (iv) $r > c$.

c. Determine the electric potential V at the center of the sphere using infinity ($r = \infty$) as a reference point.

d. If the outer shell is grounded, what would the potential at the center of the sphere be using the same reference point as in part (c).

a) Let q_b be the induced charge on inner surface. $\vec{E} = \vec{0}$ inside conductors.



For $b < r < c$, by Gauss' law then:

$$Q_{enclosed} = 0$$

$$\Rightarrow \quad q_{in} + q_b = 0$$

$q_{in} \rightarrow$ charge of insulator

$q_b \rightarrow$ charge of inner surface of conductor

$$q_{in} = \int d^3r\, \rho(r)$$

$$= \int_{S^2} d\Omega \int_0^a dr\, r^2 \rho(r)$$

$$= 4\pi \int_0^a dr\, \kappa r$$

$$= 4\pi\kappa \left[\frac{r^2}{2}\right]_0^a$$

$$q_{in} = 2\pi\kappa a^2 \Rightarrow \quad q_b = -2\pi\kappa a^2$$

Thus, $\sigma_b = \frac{-2\pi\kappa a^2}{4\pi b^2}$

$$\boxed{\sigma_b = -\frac{\kappa a^2}{2b^2}}$$

Since the conducting shell has no net charge and (ideal) conductors only have charges on their surfaces, then:

$$q_b + q_c = 0$$

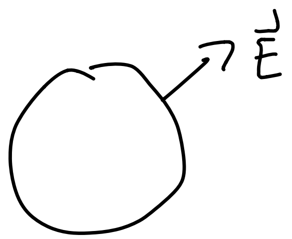
$q_c \rightarrow$ charge of outer surface

Then,

$$q_c = -q_b = 2\pi k a^2$$

$$\Rightarrow \boxed{\sigma_c = \frac{k a^2}{2 b^2}}$$

b) $r < a$



$$\oint \vec{E} \cdot d\vec{a} = Q_{enc.}/\varepsilon_0$$

$$E(4\pi r^2) = \frac{1}{\varepsilon_0} \int d\Omega \int_0^r dr' \, r'^2 \left(\frac{k}{r'}\right)$$

$$E(\cancel{4\pi} r^2) = \frac{\cancel{4\pi} k}{\varepsilon_0} \left[\frac{r'^2}{2}\right]_0^r$$

$$E \cancel{r^2} = \frac{k}{2\varepsilon_0} \cancel{r^2}$$

$$\Rightarrow \boxed{\vec{E} = \frac{k}{2\varepsilon_0} \hat{r}}$$

$a < r < b$

$$\oint \vec{E} \cdot d\vec{a} = q_{in}/\varepsilon_0$$

$$E(4\cancel{\pi} r^2) = \frac{2\cancel{\pi} k a^2}{\varepsilon_0}$$

$$\boxed{\vec{E} = \frac{K}{2\varepsilon_0}\frac{a^2}{r^2}\hat{r}}$$

$b<r<c$

$$\boxed{\vec{E} = \vec{0}}$$

(inside conductor)

$c<r$

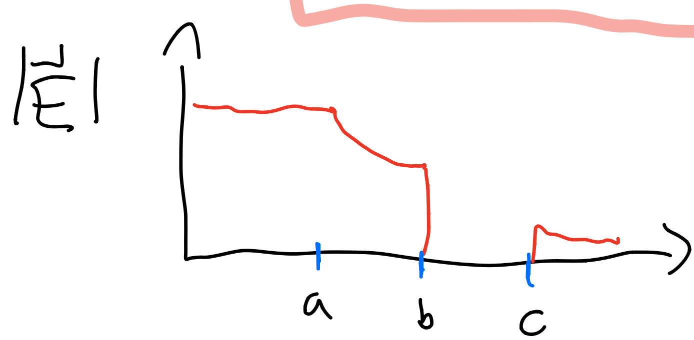
$$\oint E\cdot dA = \frac{1}{\varepsilon_0}\left(q_{in} + \underbrace{q_b + q_c}_{=0}\right)$$

$=0$, no net charge for the conductor

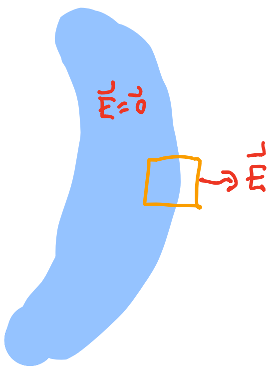
$$E(4\pi r^2) = \frac{q_{in}}{\varepsilon_0}$$

$$E(4\pi r^2) = \frac{2\pi K a^2}{\varepsilon_0}$$

$$\boxed{\vec{E} = \frac{K}{2\varepsilon_0}\frac{a^2}{r^2}\hat{r}}$$



Note: drawing a small box at the outer surface of the conductor



$$E A = \frac{\sigma A}{\epsilon_0} \Rightarrow \sigma = \frac{k a^2}{2 b^2}$$

consistent with our result from (a).

c) $$V_0 = -\int_\infty^0 d\vec{l}\cdot\vec{E} = -\int_\infty^0 dr\, E(r)$$

$$V_0 = -\int_\infty^c dr \frac{k}{2\epsilon_0}\frac{a^2}{r^2} - \int_b^a dr \frac{k}{2\epsilon_0}\frac{a^2}{r^2} - \int_a^0 dr \frac{k}{2\epsilon_0}$$

$$V_0 = \frac{k}{2\epsilon_0}\left(\left[\frac{a^2}{r}\right]_\infty^c + \left[\frac{a^2}{r}\right]_b^a - \left[r \right]_a^0 \right)$$

$$= \frac{k}{2\epsilon_0}\left(\frac{a^2}{c} - 0 + \frac{a^2}{a} - \frac{a^2}{b} - 0 + a \right)$$

$$\boxed{V_0 = \frac{k a^2}{2\epsilon_0}\left(\frac{1}{c} - \frac{1}{b} + \frac{2}{a}\right)}$$