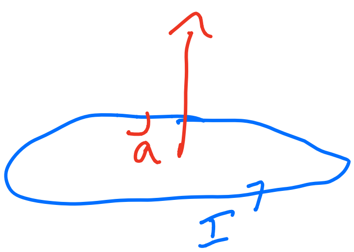
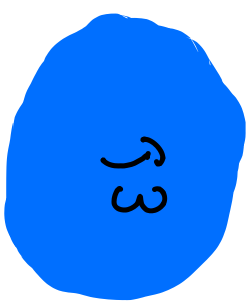
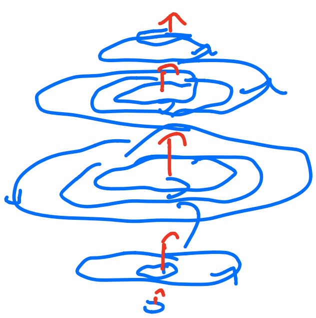
EM 3: A sphere of radius a, has a volume charge density $\rho = Ar^2$ where A is a constant. It rotates about a diameter coinciding with the z-axis with constant angular velocity, ω. What is its magnetic dipole moment?

For a ring with current, I, and an enclosed area, a, pointing in the perpendicular direction, $\vec{a}$, (use RH rule) then the magnetic dipole moment is:



$$\vec{m} = I\vec{a}$$

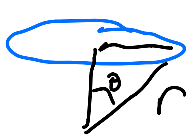




To get $\vec{m}$ for sphere, break into infinitesimal rings and integrate.

$$d\vec{m} = dI\,\vec{a} \qquad (1)$$

Now, $\vec{a} = \pi(r\sin\theta)^2\,\hat{z}$



$$\vec{a} = \pi r^2 \sin^2\theta\,\hat{z} \qquad (2)$$

Also, $$dI = \frac{dQ}{T}$$

A charge of $dQ = \rho(r)\, dV$ passes through the "wire" in $T = \frac{2\pi}{\omega}$ time.

$$dI = \frac{\rho(r)\, dV}{\frac{2\pi}{\omega}} = \frac{\omega}{2\pi} \rho(r) r^2 dr\, d\cos\theta\, d\phi \quad (3)$$

Then, putting (2) and (3) into (1):

$$d\vec{m} = \pi r^2 \sin^2\theta\, \hat{z}\, \frac{\omega}{2\pi} \rho(r)\, r^2 dr\, d\cos\theta\, d\phi$$

$$= \frac{\omega}{2} \hat{z} \rho(r) r^4 \underbrace{\sin^2\theta}_{= (1-\cos^2\theta)}\, dr\, d\cos\theta\, d\phi$$

$$d\vec{m} = \frac{\omega}{2} \hat{z} \rho(r) r^4 (1-\cos^2\theta)\, dr\, d\cos\theta\, d\phi$$

Therefore the total magnetic dipole is

$$\vec{m} = \int_{\text{sphere}} d\vec{m}$$

$$\vec{m} = \frac{\omega}{2}\hat{z} \int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta \,(1-\cos^2\theta) \int_0^a dr\, r^4 \rho(r)$$

Now, $\int_{-1}^{1} d\cos\theta\,(1-\cos^2\theta) = 2\int_0^1 d\cos\theta\,(1-\cos^2\theta)$ (Integrand even)

$$= 2\left[\cos\theta - \frac{\cos^3\theta}{3}\right]_0^1$$

$$= 2\left(1-\frac{1}{3}\right) = 4/3$$

Then,

$$\vec{m} = \frac{\omega}{\cancel{2}}\hat{z}\,(\cancel{2}\pi)\left(\frac{4}{3}\right)\int_0^a dr\, r^4 \rho(r)$$

$$\vec{m} = \frac{4\pi\omega}{3}\hat{z}\int_0^a dr\, r^4 \rho(r) \qquad (4)$$

Inserting $\rho(r) = Ar^2$ into Eq. (4):

$$\vec{m} = \frac{4\pi\omega}{3}\hat{z}\, A\int_0^a dr\, r^6$$

$$= \frac{4\pi\omega A}{3}\hat{z}\left[\frac{r^7}{7}\right]_0^a$$

$$\boxed{\vec{m} = \frac{4\pi\omega A a^7}{21}}$$

Note: inserting $\rho(r) = \sigma\,\delta(r-R)$ into Eq.(4) gives $\vec{m} = \frac{4\pi}{3}\omega\sigma R^4\,\hat{z}$, which is the answer for 2011 Classical 6(b), finding the magnetic dipole moment of uniformly distributed charge on the surface of a sphere.

For a uniform sphere $\rho(r) = \dfrac{Q}{\frac{4\pi}{3}R^3}$

and $$\vec{m} = \frac{4\pi\omega}{3}\,\hat{z}\,\frac{Q}{\frac{4\pi}{3}R^3}\int_0^R dr\, r^4$$

$$\vec{m} = \frac{1}{5}\omega Q R^2\,\hat{z}$$ (Uniform sphere)