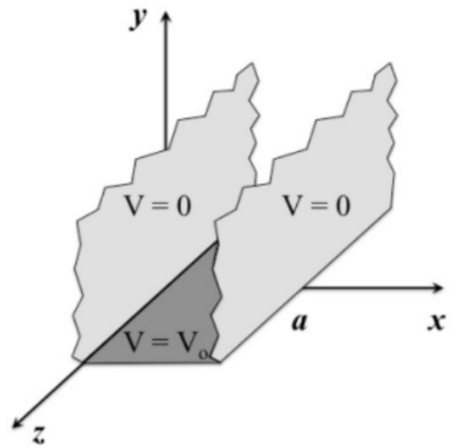
2014 Classical 4

EM4: Two infinitely large metal plates lie parallel to the *y-z* plane, one at $x = 0$ and one at $x = a$, as shown in the figure to the right. These two plates are maintained at zero potential ($V = 0$) and both extend from $y = 0$ to $y = +\infty$ and from $z = -\infty$ to $z = +\infty$. A third plate, this one maintained at a constant potential V_0, lies in the *x-z* plane and forms the bottom of a "slot". Determine an expression for the potential $V(x,y)$ for any point within the "slot". Notice that due to symmetry, the potential is independent of *z*.



Griffiths, EM (4 ed.) Example 3.3

$$V = V(x, y)$$

$$\nabla^2 V(x, y) = 0$$

$$\frac{\partial^2}{\partial x^2} V(x, y) + \frac{\partial^2}{\partial y^2} V(x, y) = 0$$

Try $V(x, y) = X(x)\,Y(y)$

$$\frac{\partial^2}{\partial x^2}\left(X(x)\,Y(y)\right) + \frac{\partial^2}{\partial y^2}\left(X(x)\,Y(y)\right) = 0$$

$$Y(y)\frac{\partial^2}{\partial x^2}X(x) + X(x)\frac{\partial^2}{\partial y^2}Y(y) = 0$$

$$Y(y)\frac{d^2X(x)}{dx^2} = -X(x)\frac{d^2Y(y)}{dy^2}$$

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = -\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} \qquad (1)$$

The LHS of Eq. (1) depends on x, the RHS on y. This is only possible if both are constant:

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = \lambda \qquad (2)$$

$$-\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = \lambda \qquad (3)$$

Now we have the following boundary conditions:

$$V(0,y) = V(a,y) = 0 \qquad \text{BC 1 a \& b}$$

$$V(x,0) = V_0 \qquad \text{BC 2}$$

Now,

$\lambda \neq 0$

Why? First assume $V_0 \neq 0$. Then the solution $V(x,y) = 0$ is not a solution since it cannot satisfy BC2.

Assume $\lambda = 0$.

From (2),

$$X(x) = mx + b$$

This only satisfies BC1 a and b if

$$X(x) = 0$$

$$\Rightarrow V(x,y) = X(x)\,Y(y) = 0,$$

which is not allowed.

λ cannot be positive

Eq. (2) becomes:

$$\frac{\partial^2 X(x)}{\partial x^2} = X(x)$$

$$\Rightarrow \quad X(x) = c_1 \cosh(\sqrt{\lambda}x) + c_2 \sinh(\sqrt{\lambda}x)$$

From BC 1a:

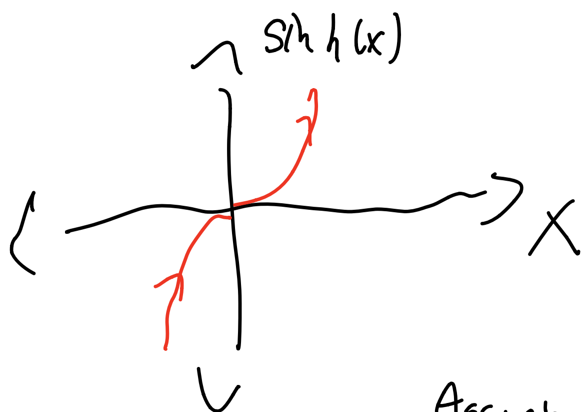
$$X(0) = 0 = c_1 \cosh(0) + c_2 \sinh(0)$$

$$0 = c_1$$

$$\Rightarrow \quad X(x) = c_2 \sinh(\sqrt{\lambda}\, x)$$

From BC 1b:

$$X(a) = 0 = c_2 \sinh(\sqrt{\lambda}\, a)$$



sinh(x) only crosses the x-axis at $x=0$

$$\Rightarrow \sqrt{\lambda}\, a = 0$$

Assuming $a \neq 0$, not possible.

<u>λ must be negative</u>

Let $\lambda = -k^2$

$$\Rightarrow \quad \frac{d^2 X(x)}{dx^2} = -k^2 X(x)$$

$$\Rightarrow \qquad X(x) = c_1 \cos(kx) + c_2 \sin(kx)$$

BC 1a:

$$X(0) = 0 = c_1 + c_2\,(0)$$

$$\Rightarrow c_1 = 0$$

$$X(x) = c_2 \sin(kx)$$

BC 1a:

$$X(a) = 0 = c_2 \sin(ka)$$

This implies:

$$ka = \pi n \qquad (n \text{ integer})$$

Therefore,

$$X(x) = c_2 \sin\left(\frac{\pi n x}{a}\right)$$

As for Eq. (3)

$$-\frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} = -k^2$$

$$\frac{d^2 Y(y)}{dy^2} = k^2 Y(y)$$

Instead of sinh and cosh it', more convenient to use exponentials:

$$Y(y) = d_1 e^{ky} + d_2 e^{-ky} \qquad (4)$$

Now we impose a third boundary condition:

$$\lim_{y \to \infty} V(x, y) < \infty$$

This is motivated by physicality. We don't want ∞ potentials.

In Eq. (4) we must then discard the e^{ky} solution for $n > 0$.

$$Y(y) = d_2 e^{-\frac{\pi n}{a} x} \qquad (5)$$

For $n < 0$, we we discard e^{-ky} but we up with $e^{-\frac{\pi |n|}{a} x}$, which is the same as Eq. (5)

$n = 0$ is not permited as $\sin(0) = 0$ and $V(x, y) = 0$ is not a solution.

These solutions can be indexed by positive n:

$$V_n(x,y) = e^{-k_n y} \sin(k_n x)$$

where $k_n = \frac{n\pi}{a}$.

The general solution will be a linear combination of these:

$$V(x,y) = \sum_{n=1}^{\infty} a_n e^{-k_n y} \sin(k_n x)$$

We now use BC2 to get the a_n's.

$$V(x,0) = V_0 = \sum_{n=1}^{\infty} a_n \sin(k_n x) \quad (6)$$

We now use the orthogonality of sine functions:

$$\int_0^L dx \, \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) = \frac{L\,\delta_{mn}}{2}$$

Thus, we multiply Eq. (G) by $\sin\left(\frac{m\pi x}{a}\right)$ and integrate from 0 to a.

$$V_0 \int_0^a dx \, \sin\left(\frac{m\pi x}{a}\right) = \sum_{n=1}^{\infty} a_n \int_0^a dx \, \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi x}{a}\right)$$

$$V_0 \left[-\frac{a}{m\pi} \cos\left(\frac{m\pi x}{a}\right)\right]_0^a = \sum_{n=1}^{\infty} a_n \frac{a\,\delta_{mn}}{2}$$

$$-\frac{V_0 \cancel{a}}{m\pi} \left[\cos(m\pi) - 1\right] = \frac{a_m \cancel{a}}{2}$$

$$\Rightarrow \quad a_m = -\frac{2V_0}{m\pi} \left[\cos(m\pi) - 1\right]$$

But $\cos(m\pi) = (-1)^m$

$$a_m = \begin{cases} \dfrac{4V_0}{m\pi} & m \text{ odd} \\ 0 & m \text{ even} \end{cases}$$

Therefore,

$$\boxed{V(x,y) = \frac{4V_0}{\pi} \sum_{\ell=0}^{\infty} \frac{e^{-\frac{(2\ell+1)\pi y}{a}} \sin\left(\frac{(2\ell+1)\pi}{a} x\right)}{2\ell+1}}$$

Bonus: Above is sufficient for full points.
Further simplification below is for fun.

Use $\sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix})$

$$V(x,y) = \frac{4V}{\pi}\sum_{l=0}^{\infty}\frac{1}{2i(2l+1)}\left[e^{-\frac{(2l+1)\pi}{a}(y-ix)} - e^{-\frac{(2l+1)\pi}{a}(y+ix)}\right]$$

(7)

Side note: $\left|e^{-\frac{(2l+1)\pi}{a}(y\pm ix)}\right| \leq 1$

since $y \geq 0$

$\therefore$ series converges

Now let

$$f(\alpha) = \sum_{l=0}^{\infty}\frac{e^{(2l+1)\alpha}}{(2l+1)}$$

$$f'(\alpha) = \sum_{l=0}^{\infty} e^{(2l+1)\alpha}$$

$$= e^{\alpha}\sum_{l=0}^{\infty}(e^{2\alpha})^{l}$$

$$f'(\alpha) = e^{\alpha}\frac{1}{1-e^{2\alpha}}$$

$$f(\alpha) = \int_0^{\alpha} d\alpha' \frac{e^{\alpha'}}{1-e^{2\alpha'}}$$

Look up integral

$$f(\alpha) = \frac{1}{2} \ln\left|\frac{1+\alpha}{1-\alpha}\right| \quad (-1<\alpha<1)$$

Thus,

$$V(x,y) = \frac{4V_0}{\pi} \sum_{l=0}^{\infty} \frac{1}{2i(2l+1)} \left[e^{-\frac{(2l+1)\pi}{a}(y-ix)} - e^{-\frac{(2l+1)\pi}{a}(y+ix)} \right]$$

$$V(x,y) = \frac{V_0}{i\pi} \left[\ln\left|\frac{1+e^{-\frac{\pi}{a}(y-ix)}}{1-e^{-\frac{\pi}{a}(y-ix)}}\right| - \ln\left|\frac{1+e^{-\frac{\pi}{a}(y+ix)}}{1-e^{-\frac{\pi}{a}(y-ix)}}\right| \right]$$

We can use $\frac{1+e^{-\alpha}}{1-e^{-\alpha}} = \frac{e^{\alpha/2}+e^{-\alpha/2}}{e^{\alpha/2}-e^{-\alpha/2}} = \frac{\cosh(\frac{\alpha}{2})}{\sinh(\frac{\alpha}{2})} = \tanh(\frac{\alpha}{2})$

$$= \frac{V_0}{i\pi} \ln\left(\frac{\tanh\left(\frac{\pi}{2a}(y+ix)\right)}{\tanh\left(\frac{\pi}{2a}(y-ix)\right)} \right)$$

Now we $\tan(a+ib) = \frac{\sinh(2a) + i\sinh(2b)}{\cosh(2a) + i\cos(2b)}$

$$= \frac{V_0}{i\pi} \ln\left(\frac{\sinh\left(\frac{\pi y}{a}\right) + i \sin\left(\frac{\pi}{a} x\right)}{\sinh\left(\frac{\pi y}{a}\right) - i \sin\left(\frac{\pi}{a} x\right)} \right)$$

$$= \frac{2V_0}{\pi} \left[\frac{-i}{2} \ln\left(\frac{\frac{\sinh\left(\frac{\pi y}{a}\right)}{\sin\left(\frac{\pi x}{a}\right)} + i}{\frac{\sinh(\pi y/a)}{\sin(\pi x/a)} - i} \right) \right]$$

but $\tan^{-1}(z) = \frac{-i}{2} \ln\left(\frac{z+i}{z-i}\right)$

$$\boxed{V(x, y) = \frac{2V_0}{\pi} \tan^{-1}\left(\frac{\sinh(\pi y/a)}{\sin(\pi x/a)} \right)}$$

Note: Griffiths displays above equation, doesn't derive it.