SP1: Two state paramagnet: Consider a solid made up of a collection of magnetic dipoles. Each dipole can be in one of two states: antiparallel to the applied magnetic field and parallel to the applied magnetic field. Let ϵ and $-\epsilon$ be their respective energies (ϵ can be positive or negative). Let the solid be made up of N dipoles.

(a) Derive an expression for the number of microstates in the macrostate with $N_\uparrow$ dipoles aligned parallel to the applied magnetic field.

(b) Write down the expression for the total energy U in terms of N and $N_\uparrow$.

(c) Assume $N, N_\uparrow, (N - N_\uparrow)$ are all large compared to unity. Prove that

$$U = -N\epsilon \tanh \frac{\epsilon}{kT}.$$

a) $$\Omega = \binom{N}{N_\uparrow} = \frac{N!}{N_\uparrow!\,(N-N_\uparrow)!}$$

$$\ln \Omega = \ln N! - \ln N_\uparrow! - \ln (N-N_\uparrow)!$$

Use Stirling's approx: $\ln M! \approx M \ln M - M$

$$\ln \Omega! = N \ln N - \cancel{N} - N_\uparrow \ln N_\uparrow + \cancel{N_\uparrow}$$

$$- (N-N_\uparrow) \ln (N-N_\uparrow) + \cancel{N} - \cancel{N_\uparrow}$$

$$\ln \Omega = N \ln N - N_\uparrow \ln N_\uparrow - (N-N_\uparrow) \ln (N-N_\uparrow)$$

$$\ln \Omega = N\left(\ln N - \ln(N - N_\uparrow)\right) + N_\uparrow\left(\ln(N - N_\uparrow) - \ln N_\uparrow\right)$$

$$\boxed{\ln \Omega = N \ln\left(\frac{N}{N - N_\uparrow}\right) + N_\uparrow \ln\left(\frac{N - N_\uparrow}{N_\uparrow}\right)}$$

b)

$$U = -\varepsilon N_\uparrow + N_\downarrow \varepsilon$$

$$= -\varepsilon N_\uparrow + \varepsilon (N - N_\uparrow)$$

$$\boxed{U = \varepsilon (N - 2N_\uparrow)}$$

c) For a single particle:

$$z = \sum_n e^{-E_n \beta}$$

$$z = e^{-\varepsilon\beta} + e^{\varepsilon\beta}$$

For n independent particles the total partition function is

$$Z = \prod_{l=1}^{N} z_l$$

$$Z = z^N$$

$$Z = (e^{-\varepsilon\beta} + e^{\varepsilon\beta})^N$$

The total energy is

$$U = -\frac{\partial}{\partial\beta} \log Z$$

$$U = -N \frac{\partial}{\partial \beta} \log\left(e^{-\varepsilon\beta} + e^{\varepsilon\beta}\right)$$

$$U = -N \left(\frac{-\varepsilon e^{-\varepsilon\beta} + \varepsilon e^{\varepsilon\beta}}{e^{-\varepsilon\beta} + e^{\varepsilon\beta}}\right)$$

$$U = -N\varepsilon \left(\frac{-e^{-\varepsilon\beta} + e^{\varepsilon\beta}}{e^{-\varepsilon\beta} + e^{\varepsilon\beta}}\right)$$

$$U = -N\varepsilon \frac{\sinh(\varepsilon\beta)}{\cosh(\varepsilon\beta)}$$

$$\boxed{U = -N\varepsilon \tanh(\varepsilon\beta)}$$