MP1: Let $\mathbf{u} = d\mathbf{r}/dt$ be the velocity of a particle observed in an inertial frame K. The same quantity observed in an inertial frame K' moving with velocity $\mathbf{v}$ with respect to K is $\mathbf{u}' = d\mathbf{r}'/dt$.

a) Use the transformation properties of dt, $\mathbf{r}_\parallel$ and $\mathbf{r}_\perp$(the direction along the velocity $\mathbf{v}$ or perpendicular to the moving K' frame, respectively), to directly derive the velocity addition rule,

$$\mathbf{u}_\parallel = \frac{d\mathbf{r}_\parallel}{dt} = \frac{\mathbf{u}'_\parallel + \mathbf{v}}{1 + \dfrac{\mathbf{v}\cdot\mathbf{u}'_\parallel}{c^2}} \quad \text{and} \quad \mathbf{u}_\perp = \frac{d\mathbf{r}_\perp}{dt} = \frac{\mathbf{u}'_\perp}{\gamma(v)\left[1 + \dfrac{\mathbf{v}\cdot\mathbf{u}'_\parallel}{c^2}\right]}.$$

b) Let $\mathbf{v}$ define a polar axis with polar coordinates $\mathbf{u} = (u, \theta)$, and $\mathbf{u}' = (u', \theta')$ for the particle velocities as measured in K and K'. Write the transformation laws in part (a) in the form of $u = u(u', \theta')$ and $\theta = \theta(u', \theta')$

c) Use the results of (b) to show that $u \to c$ when $v \to c$.

a) We can choose our co-ordinates so that the $\parallel$ direction is x and $\perp$ is the yz plane.

A Lorentz transformation relates K and K':

$$dt = \gamma\left(dt' + v\,dx'/c^2\right) \qquad (1)$$
$$dx = \gamma\left(dx' + v\,dt'\right) \qquad (2)$$
$$dy = dy' \qquad (3)$$
$$dz = dz' \qquad (4)$$

By Eq. (1) and (2),

$$u_x = \frac{dx}{dt} = \frac{\gamma\left(dx' + v\,dt'\right)}{\gamma\left(dt' + v\,dx'/c^2\right)}$$

$$u_x = \frac{dx' + v\,dt'}{dt' + \frac{v\,dx'}{c^2}} \qquad \frac{1/dt'}{1/dt'}$$

$$u_x = \frac{u_x' + V}{1 + \frac{v u_x'}{c^2}}$$

Now, since $\vec{V} = V\hat{x}$ then

$$\vec{V}\cdot\vec{u}' = V u_x'$$

So,

$$u_x = \frac{u_x' + V}{1 + \frac{\vec{V}\cdot\vec{u}_x'}{c^2}}$$

$$\Rightarrow \boxed{\vec{u}_{\parallel} = \frac{\vec{u}_{\parallel}' + \vec{V}}{1 + \frac{\vec{V}\cdot\vec{u}_{\parallel}'}{c^2}}} \qquad (5)$$

Now, from Eq. (1) and (3),

$$u_y = \frac{dy}{dt} = \frac{dy'}{\gamma\left(dt' + \frac{v\,dx'}{c^2}\right)}$$

$\frac{1/dt'}{1/dt'}$

$$u_y = \frac{u_y'}{\gamma(1 + v u_x'/c^2)}$$

use logic from earlier for $v u_x'$

$$u_y = \frac{u_y'}{\gamma(1 + \vec{v}\cdot\vec{u}_\parallel'/c^2)}$$

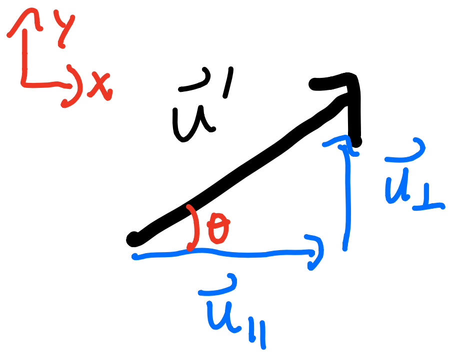
Logic for u_z is the same:

$$u_z = \frac{u_z'}{\gamma(1 + \vec{v}\cdot\vec{u}_\parallel'/c^2)}$$

$$\Rightarrow \quad \boxed{\vec{u}_\perp = \frac{\vec{u}_\perp'}{\gamma(1 + \vec{v}\cdot\vec{u}_\parallel'/c^2)}} \quad (6)$$

b) Choose co-ordinates so that $\vec{u}_\perp$ is in the $\hat{y}$ direction.

Then, from Eq. (5) and (6)



$$\tan\theta = \frac{u_y}{u_x} = \frac{\dfrac{u_y'}{\gamma(V)\cancel{(1+\vec{V}\cdot\vec{u}_\parallel/c^2)}}}{\dfrac{u_x' + V}{\cancel{(1+\vec{V}\cdot\vec{u}_\parallel/c^2)}}}$$

$$\tan\theta = \frac{u_y'}{\gamma(u_x' + V)} = \frac{u'\sin\theta'}{\gamma(u'\cos\theta' + V)} = \frac{u'\tan\theta'}{\gamma\left(u' + \dfrac{V}{\cos\theta'}\right)}$$

$$\Rightarrow \boxed{\theta = \tan^{-1}\left[\frac{u'\tan\theta'}{\gamma(u' + V/\cos\theta)}\right]} \qquad (7)$$

Note: as $V \to 0$ $\theta \to \theta'$ as expected.

Now,

$$u^2 = u_\perp^2 + u_\parallel^2$$

Use Eq.(5) and (6)

$$= \frac{u_y'^2}{\gamma^2\left(1+\frac{u_x' V}{c^2}\right)^2} + \frac{(u_x' + V)^2}{\left(1+\frac{u_x V}{c^2}\right)^2}$$

$$u^2 = \frac{1}{\left(1+\frac{u_x' v}{c^2}\right)^2}\left[u_y'^2 \left(1 - v^2/c^2\right) + \left(u_x' + v\right)^2 \right]$$

$$= \frac{1}{\left(1+\frac{u'\cos\theta' v}{c^2}\right)^2}\left[u'^2 \sin^2\theta' \left(1 - v^2/c^2\right) + \left(u'\cos\theta' + v\right)^2 \right.$$

$$= \frac{1}{\left(1+\frac{u'\cos\theta' v}{c^2}\right)^2}\left[u'^2 \sin^2\theta' \left(1 - \frac{v^2}{c^2}\right) + u'^2\cos^2\theta' + 2u'\cos\theta' v + v^2 \right]$$

$$= \frac{1}{\left(1+\frac{u'\cos\theta' v}{c^2}\right)^2}\left[u'^2 - \frac{u'^2 v'^2}{c^2}\sin^2\theta' + 2u' v\cos\theta' + v^2 \right]$$

$$\Rightarrow \boxed{u' = \frac{\sqrt{u'^2 - \frac{u'^2 v'^2}{c^2}\sin^2\theta' + 2u' v\cos\theta' + v^2}}{1 + \frac{u'\cos\theta' v}{c^2}}} \quad (8)$$

Sanity check: as $v \to 0$, $u \to u'$

c) Taking $v \to c$ in Eq. (8),

$$u' \to \frac{\sqrt{u'^2 - u'^2 \sin^2\theta' + 2u'c\cos\theta' + c^2}}{1 + \frac{u'\cos\theta'}{c}}$$

Use $\sin^2\theta = 1 - \cos^2\theta$

$$u' \to \frac{\sqrt{u'^2 - u'^2(1-\cos^2\theta') + 2u'c\cos\theta' + c^2}}{1 + \frac{u'\cos\theta'}{c}}$$

$$= \frac{\sqrt{u'^2\cos^2\theta' + 2u'c\cos\theta' + c^2}}{1 + \frac{u'\cos\theta'}{c}}$$

$$= \frac{\sqrt{(u'\cos\theta' + c)^2}}{1 + \frac{u'\cos\theta'}{c}}$$

$$= \frac{u'\cos\theta' + c}{1 + \frac{u'\cos\theta'}{c}}$$

$$= c\,\frac{(1 + u\cos\theta'/c)}{(1 + u'\cos\theta'/c)}$$

$$\therefore \boxed{u \to c \quad \text{as} \quad v \to c}$$