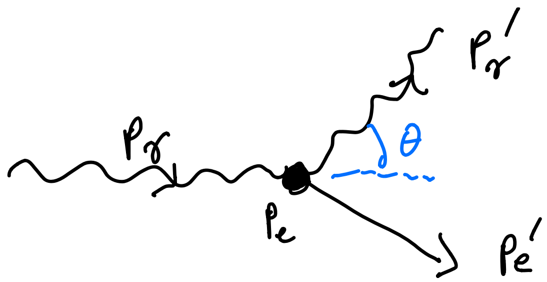
MP4: Consider the process of Compton scattering. A photon of wavelength λ is scattered off a free electron initially at rest. Let λ' be the wavelength of the photon scattered in a direction θ relative to the photon incident direction.

(a) Find λ' in terms of λ and θ and universal constants.

(b) Find the kinetic energy of the recoiled electron.



By conservation of 4-momentum:

$$p_\gamma^\mu + p_e^\mu = p_\gamma'^\mu + p_e'^\mu$$

$$p_\gamma^\mu - p_\gamma'^\mu = p_e'^\mu - p_e^\mu$$

(Square both sides)

$$(p_\gamma^\mu - p_\gamma'^\mu)^2 = (p_e'^\mu - p_e^\mu)^2$$

$$\underbrace{p_\gamma^2}_{=c^2 m_\gamma^2 = 0} - 2p_\gamma \cdot p_\gamma' + \underbrace{p_\gamma'^2}_{=m_\gamma^2 c^2 = 0} = \underbrace{p_e'^2}_{=m_e^2 c^2} - 2p_e' \cdot p_e + \underbrace{p_e^2}_{=m_e^2 c^2}$$

$$-2p_\gamma \cdot p_\gamma' = m_e^2 c^2 - 2p_e' \cdot p_e + m_e^2 c^2$$

$$-\cancel{2}\left(\frac{E_\gamma}{c}\frac{E_\gamma'}{c}-|\vec{p}_\gamma||\vec{p}_\gamma'|\cos\theta_\gamma\right)=\cancel{2}m_e^2c^2-\cancel{2}\left(\frac{E_eE_e'}{cc}-|\vec{p}_e||\vec{p}_e'|\cos\theta_e\right)$$

Now, because the photon is massless

$$\frac{E_\gamma}{c}=|\vec{p}_\gamma| \quad \text{and} \quad E_\gamma'=|\vec{p}_\gamma'|$$

Since the electron is initially at rest,

$$E_e=m_ec^2 \quad \text{and} \quad |\vec{p}_e|=0$$

Thus,

$$-\left(\frac{E_\gamma}{c}\frac{E_\gamma'}{c}-\frac{E_\gamma}{c}\frac{E_\gamma'}{c}\cos\theta_\gamma\right)=m_e^2c^2-\left(\frac{m_ec^2}{c}\frac{E_e'}{c}-0\right)$$

$$-\frac{E_\gamma E_\gamma'}{c^2}(1-\cos\theta_\gamma)=m_e^2c^2-m_eE_e' \qquad (1)$$

Now, by conservation of energy,

$$E_\gamma+m_ec^2=E_\gamma'+E_e'$$

$$\Rightarrow \quad E_e'=E_\gamma-E_\gamma'+m_ec^2 \qquad (2)$$

Insert Eq. (2) into Eq. (1),

$$-\frac{E_\gamma E_\gamma'}{c^2}(1-\cos\theta_\gamma) = m_e^2c^2 - m_e(E_\gamma - E_\gamma' + m_e c^2)$$

$$-\frac{E_\gamma E_\gamma'}{c^2}(1-\cos\theta_\gamma) = \cancel{m_e^2c^2} - E_\gamma + E_\gamma' - \cancel{m_e^2c^2}$$

$$-\frac{E_\gamma E_\gamma'}{c^2}(1-\cos\theta_\gamma) = (-E_\gamma + E_\gamma')m_e$$

$$\frac{E_\gamma E_\gamma'}{c^2}(1-\cos\theta_\gamma) = (E_\gamma - E_\gamma')m_e$$

Now, insert

$$E = hf = \frac{hc}{\lambda} \qquad (3)$$

into Eq. (3):

$$\frac{h^{\cancel{2}}c^{\cancel{2}}}{\lambda\lambda'c^2}(1-\cos\theta_\gamma) = \left(\frac{\cancel{hc}}{\lambda} - \frac{\cancel{hc}}{\lambda'}\right)m_e$$

$$h\cancel{c}(1-\cos\theta_\gamma) = \lambda\lambda'c^{\cancel{2}}\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right)m_e$$

$$\boxed{\frac{h}{m_e c}(1-\cos\theta_\gamma) = \lambda' - \lambda}$$

b) $KE_e' = E_e' - m_e c^2$

Using Eq. (2)

$$E_e' = E_\gamma - E_\gamma' + m_e c^2$$

$$KE_e' = E_\gamma - E_\gamma' + \cancel{m_e c^2} - \cancel{m_e c^2}$$

$$KE_e' = \frac{hc}{\lambda} - \frac{hc}{\lambda'}$$

$$\boxed{KE_e' = hc\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right)}$$