Modern 2021 - QM2

QM1: An electron spin is in a uniform magnetic field $\vec{B} = B_0\,\hat{j}$ (where $\hat{j}$ is the unit vector in the $+y$ direction). At $t = 0$, the electron spin is aligned in the $+z$ direction. Find the spin wavefunction at late time t. Express your answer in terms of the $|-\rangle$ and $|+\rangle$ eigenstates in the z direction.

Method 1

$$\hat{H} = -\vec{\mu}\cdot\vec{B}$$

Now,

$$\vec{\mu} = \gamma\frac{\hbar}{2}\vec{\sigma}$$

where $\gamma = g\frac{e}{2m}$. Using $g = 2$ (good approx.) and $q = -e$ for electron.

Then, $\gamma = \frac{-e}{m}$ and

$$\hat{H} = \frac{e\hbar}{2m}\vec{B}\cdot\vec{\sigma}$$

The Schrödinger equation is then

$$i\hbar\frac{\partial\psi(t)}{\partial t} = \frac{e}{2m}\hbar\,\vec{B}\cdot\vec{\sigma}\,\psi(t)$$

$$\Rightarrow \quad i\frac{\partial\psi(t)}{\partial t} = \frac{e}{2m}\vec{B}\cdot\vec{\sigma}\,\psi(t)$$

inserting $\vec{B} = B\hat{j}$,

$$i \frac{\partial \psi(t)}{\partial t} = \frac{eB}{2m} \sigma_y \psi(t)$$

$$\frac{\partial \psi(t)}{\partial t} = \frac{-ieB}{2m} \sigma_y \psi(t)$$

Let $\omega = \frac{eB}{m}$

$$\frac{\partial \psi(t)}{\partial t} = \frac{-i\omega}{2} \sigma_y \psi(t) \qquad (1)$$

Using the fact that σ_y has eigen-values ± 1, then the solution to Eq. (1) is (in bra-ket notation),

$$|\psi(t)\rangle = c_1 e^{\frac{-i\omega t}{2}} |+\rangle_y + c_2 e^{\frac{i\omega t}{2}} |-\rangle_y$$

Using $|+\rangle_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$, $|-\rangle_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$

in the z basis, then

$$|\psi(t)\rangle = \frac{c_1 e^{-\frac{i\omega t}{2}}}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{c_2 e^{\frac{i\omega t}{2}}}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Now, $|\psi(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{c_1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{c_2}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\Rightarrow \quad \begin{aligned} 1 &= \tfrac{1}{\sqrt{2}}(c_1 + c_2) \qquad (2) \\ 0 &= \tfrac{1}{\sqrt{2}}(c_1 - c_2) \qquad (3) \end{aligned}$$

Eq. 3 implies $c_2 = c_1$. Then,

$$1 = \frac{2c_1}{\sqrt{2}} \Rightarrow c_1 = c_2 = \frac{1}{\sqrt{2}}$$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} e^{-i\omega t/2} + \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{\frac{i\omega t}{2}}$$

$$= \frac{1}{2} \begin{pmatrix} e^{-i\omega t/2} + e^{i\omega t/2} \\ i\left(e^{-i\omega t/2} - e^{i\omega t/2}\right) \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 2\cos(\omega t/2) \\ i\left(-2i\sin(\omega t/2)\right) \end{pmatrix},$$

$$\Rightarrow \qquad |\psi(t)\rangle = \begin{pmatrix} \cos(\omega t/2) \\ \sin(\omega t/2) \end{pmatrix}$$

Or in fully bra-ket notation,

$$\boxed{|\psi(t)\rangle = \cos(\omega t/2)\,|+\rangle_z + \sin\left(\frac{\omega t}{2}\right)|-\rangle_z}$$

Method 2

We can express $\psi(t)$ as

$$\psi(t) = U(t,0)\,\psi(0)$$

where $U(t,0)$ is the time evolution operator. It can be expressed as (see 2020 Modern-QM2 solution)

$$\psi(t) = \left[I \cos\left(\frac{\omega t}{2}\right) - i\,\hat{B}\cdot\hat{\sigma}\,\sin\left(\frac{\omega t}{2}\right)\right]\psi(0)$$

where $\omega = \frac{e}{m}|\vec{B}|$.

Inserting $\hat{B} = \hat{j}$ and $\psi(0) = \begin{pmatrix}1\\0\end{pmatrix}$ (in z basis)

$$\psi(t) = \left[I \cos\left(\frac{\omega t}{2}\right) - i\,\sigma_y \sin\left(\frac{\omega t}{2}\right)\right]\begin{pmatrix}1\\0\end{pmatrix}$$

$$= \begin{pmatrix} \cos\left(\frac{\omega t}{2}\right) & -\sin\left(\frac{\omega t}{2}\right) \\ \sin(\omega t/2) & \cos\left(\frac{\omega t}{2}\right) \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

$$\psi(t) = \begin{pmatrix} \cos(\omega t/2) \\ \sin(\omega t/2) \end{pmatrix}$$

Or in bra-ket notation,

$$\boxed{|\psi(t)\rangle = \cos(\omega t/2)\,|+\rangle_z + \sin\left(\frac{\omega t}{2}\right)|-\rangle_z}$$