2009 Modern - B5

QM3: A particle of mass m and charge q is in a one-dimensional harmonic oscillator potential moving with frequency ω. In addition, it is subject to a *weak* electric field $\mathcal{E}$.

a) Find the exact expression for the energy.

b) Calculate the energy up to the first non-zero correction in non-degenerate perturbation theory. Compare your result to what you found in part a.

a) The Hamiltonian for a 1D HO is

$$\hat{H}_{HO} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2$$

Classically, the potential energy in an Electric potential is given by:

$$U = qV$$

Since $E_x = -\frac{dV}{dX}$ $\Rightarrow$ $V = -E_x X$
(chose const. of zero)

$$U = -qE_x X$$

Thus, the perturbed Hamiltonian is

$$\hat{H}' = -q\mathcal{E}\hat{x}$$

The total Hamiltonian is:

$$\hat{H} = \hat{H}_{Hb} + \hat{H}'$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 - q\mathcal{E}\hat{x} \qquad (1)$$

We can complete the square so that Eq.(1) resembles a harmonic oscillator.

$$ax^2 + bx = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a}$$

$$\Rightarrow \quad \frac{m\omega^2}{2}\hat{x}^2 - q\mathcal{E}\hat{x} = \frac{m\omega^2}{2}\left(\hat{x} + \frac{(-q\mathcal{E})}{2\left(\frac{m\omega^2}{2}\right)}\right)^2 - \frac{q^2\mathcal{E}^2}{4\frac{m\omega^2}{2}}$$

$$\frac{m\omega^2}{2}\hat{x}^2 - q\mathcal{E}\hat{x} = \frac{m\omega^2}{2}\left(\hat{x} - \frac{q\mathcal{E}}{m\omega^2}\right)^2 - \frac{q^2\mathcal{E}^2}{2m\omega^2} \qquad (2)$$

Inserting Eq. (2) into (1):

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\left(\hat{x} - \frac{q\mathcal{E}}{m\omega^2}\right)^2 - \frac{q^2\mathcal{E}^2}{2m\omega^2}$$

Letting $\hat{y} = \hat{x} - \frac{q\mathcal{E}}{m\omega^2}$ (note momentum doesn't change under spatial translations);

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{y}^2 - \frac{q^2\mathcal{E}^2}{2m\omega^2}$$

Thus, the energy is just that of a HO shifted by $-q^2\mathcal{E}^2/2m\omega^2$:

$$\boxed{E_n = \hbar\omega\left(n+\tfrac{1}{2}\right) - \frac{q^2\mathcal{E}^2}{2m\omega^2}} \quad (3)$$

b) First order

$$E_n^{(1)} = \langle n^{(0)}| H' |n^{(0)}\rangle$$

$$= -q\mathcal{E}\ \langle n|\hat{x}|n\rangle$$

Here $|n\rangle$ is the n^{th} eigenstate of a H/O.

Now: $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}^+ + \hat{a}\right)$

where $\hat{a}^+, \hat{a}$ are the creation and annihilation operators:

$$E_n^{(1)} = -q\mathcal{E}\sqrt{\frac{\hbar}{2m\omega}}\left(\langle n|\hat{a}^+|n\rangle + \langle n|\hat{a}|n\rangle\right)$$

Now, $\langle n|\hat{a}^{\dagger}|n\rangle = \sqrt{n+1}\,\langle n|n+1\rangle = 0$

$\langle n|\hat{a}|n\rangle = \sqrt{n}\,\langle n|n-1\rangle = 0$

Therefore, $\boxed{E_n^{(1)} = 0}$

This is expected as the correction in Eq. (3) is second order in ε^2.

Second order

$$E_n^{(2)} = \sum_{m\neq n} \frac{|\langle n^{(0)}|H'|m^{(0)}\rangle|^2}{E_n^{(0)} - E_m^{(0)}}$$

Now, $E_n^{(0)} - E_m^{(0)} = \hbar\omega(n+\frac{1}{2}) - \hbar\omega(m+\frac{1}{2}) = \hbar\omega(n-m)$

Also,

$\langle n^{(0)}|H'|m^{(0)}\rangle$

$= -q\varepsilon\langle n|\hat{x}|m\rangle$

$= -q\varepsilon\sqrt{\frac{\hbar}{2m\omega}}\,\langle n|\,\hat{a}^{\dagger} + \hat{a}\,|m\rangle$

$= -q\varepsilon\sqrt{\frac{\hbar}{2m\omega}}\left(\langle n|\hat{a}^{\dagger}|m\rangle + \langle n|\hat{a}|m\rangle\right)$

$$= -q\mathcal{E}\sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{m+1}\,\langle n|m+1\rangle + \sqrt{m}\,\langle n|m-1\rangle\right)$$

$$\langle n^{(0)}|H'|n^{(0)}\rangle = -q\mathcal{E}\sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{m+1}\,\delta_{n,m+1} + \sqrt{m}\,\delta_{n,m-1}\right)$$

Squaring,

$$|\langle n^{(0)}|H'|n^{(0)}\rangle|^2 = \frac{q^2\mathcal{E}^2\hbar}{2m\omega}\left((m+1)\,\delta_{n,m+1} + 2\sqrt{(m+1)m}\,\delta_{n,m+1}\delta_{n,m-1} + m\,\delta_{n,m-1}\right)$$

We've used $\delta_{m,n}^2 = \delta_{m,n}$.

Thus,

$$E_n^{(2)} = \sum_{m\neq n} \frac{q^2\mathcal{E}^2\cancel{\hbar}}{2m\omega}\left[\frac{(m+1)\,\delta_{n,m+1} + \sqrt{(m+1)m}\,\delta_{n,m-1}\delta_{n,m+1} + m\,\delta_{n,m-1}}{\cancel{\hbar}\omega\,(n-m)}\right] \qquad (4)$$

mass

index in brackets

Bad notation on my part. Don't confuse the two m's.

The middle term in the brackets disappears since it requires both $n = m-1$ and $n = m+1$.

$\delta_{n,m+1} \rightarrow$ fixes $m = n-1$

$\delta_{n,m-1} \rightarrow$ fixes $m = n+1$

Then,

$$E_n^{(2)} = \frac{q^2\varepsilon^2}{2m\omega^2}\left[\frac{n}{(n-(n-1))} + \frac{n+1}{n-(n+1)}\right]$$

$$E_n^{(2)} = \frac{q^2\varepsilon^2}{2m\omega^2}\left[\,n - (n+1)\,\right]$$

$$\Rightarrow \quad \boxed{E_n^{(2)} = -\frac{q^2\varepsilon^2}{2m\omega^2}}$$

This matches with Eq. (3), as it should.