Modern 2014 - QM4

QM4: A particle with positive energy ($E > 0$) encounters a Dirac delta-function potential barrier

$$V(x) = \alpha\delta(x)$$

where α is positive real constant with the units of energy×length.

a) Solve the Schrödinger equation for the regions $x < 0$ and $x > 0$. If you make any assumptions, you must verify your answer to receive full credit.
b) State and apply the appropriate boundary conditions for the potential interface at $x = 0$. Obtain expressions relating the different coefficients.
c) Using the results from part b), determine the transmission coefficient T for a particle incident and transmitted through this potential barrier. Discuss the limiting cases for T as the energy E approaches zero ($E \to 0$) and E approaches infinity ($E \to \infty$).

a) For $x > 0$, $V(x) = 0$ and:

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi = E\psi$$

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi$$

Let $k = \sqrt{\frac{2mE}{\hbar^2}}$. Then,

$$\frac{d^2\psi}{dx^2} = -k^2\psi \qquad (1)$$

Eq. (1) has the solution

$$\boxed{\psi_+(x) = c_1 e^{ikx} + c_2 e^{-ikx}} \qquad (2)$$

The logic for $x<0$ is identical

$$\boxed{\psi_-(x) = d_1 e^{ikx} + d_2 e^{-ikx}} \qquad (3)$$

b) We require continuity

$$\psi_+(0) = \psi_-(0) \qquad (4)$$

Inserting Eq. (2) and (3) into (4):

$$\boxed{c_1 + c_2 = d_1 + d_2} \qquad (6)$$

Now,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \alpha\delta(x)\psi = E\psi$$

Integrate both sides from $-\varepsilon$ to ε

$$-\frac{\hbar^2}{2m}\int_{-\varepsilon}^{\varepsilon} dx \frac{d^2\psi}{dx^2} + \alpha\int_{-\varepsilon}^{\varepsilon} dx\, \delta(x)\psi(x) = E\int_{-\varepsilon}^{\varepsilon} dx\, \psi(x)$$

$$-\frac{\hbar^2}{2m}\left[\frac{d\psi}{dx}\right]_{-\varepsilon}^{\varepsilon} + \alpha\psi(0) = E\int_{-\varepsilon}^{\varepsilon} dx\, \psi(x)$$

Letting $\varepsilon \to 0$,

$$-\frac{\hbar^2}{2m}\left[\frac{d\psi_+}{dx}\Big|_{x=0} - \frac{d\psi_-}{dx}\Big|_{x=0}\right] + \alpha\psi(0) = 0$$

$$\frac{d\psi_+}{dx}\Big|_{x=0} - \frac{d\psi_-}{dx}\Big|_{x=0} = \frac{2m\alpha}{\hbar^2}\psi(0)$$

From Eq. (2) and (3):

$$\boxed{ik(c_1-c_2) - ik(d_1-d_2) = \frac{2m\alpha}{\hbar^2}(c_1+c_2)} \quad (7)$$

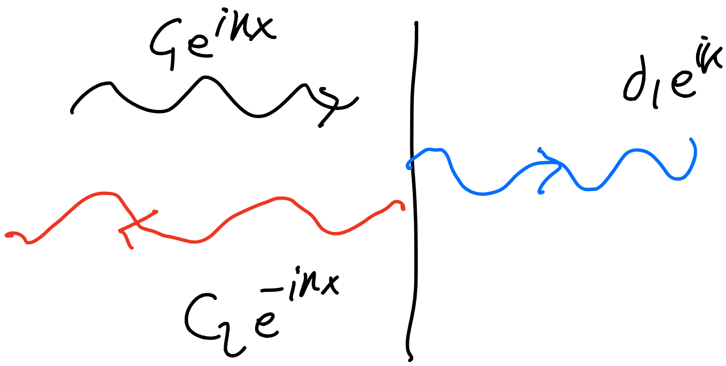
Note: since $\psi_+(0) = \psi_-(0)$ we could have used $d_1 + d_2$ on the RHS.

c) Let's take the incident particle moving in the positive x direction from $-\infty$. Then $d_2 = 0$ and Eq. (6) and (7) become:

$$c_1 + c_2 = d_1 \quad (8)$$

$$ik(c_1-c_2) + ik\,d_1 = \frac{2m\alpha}{\hbar^2}(c_1+c_2) \quad (9)$$

Since $T = \left|\frac{d_1}{c_1}\right|^2$ then we use $c_2 = d_1 - c_1$



$$ik(c_1 + c_1 - d_1) - ik d_1 = \frac{2m\alpha}{\hbar^2}(c_1 + d_1 - c_1)$$

$$\cancel{2} ik c_1 - \cancel{2} ik d_1 = \frac{\cancel{2} m\alpha}{\hbar^2} d_1$$

$$\left(-\frac{m\alpha}{\hbar^2} - ik\right) d_1 = -ik c_1$$

$$\frac{d_1}{c_1} = \frac{ik}{\frac{m\alpha}{\hbar^2} + ik} \qquad \begin{array}{l} 1/ik \\ 1/ik \end{array}$$

$$\frac{d_1}{c_1} = \frac{1}{1 - i\frac{m\alpha}{\hbar^2 k}}$$

$$\Rightarrow \quad T = \left|\frac{d_1}{c_1}\right|^2 = \frac{1}{1 + \frac{m^2\alpha^2}{\hbar^4 k^2}}$$

Using $k^2 = \frac{2mE}{\hbar^2}$,

$$T = \frac{1}{1 + \frac{m^2 \alpha^2}{\hbar^4 \left(\frac{2mE}{\hbar^2}\right)}}$$

$$\boxed{T = \frac{1}{1 + \frac{m \alpha^2}{2\hbar^2 E}}} \qquad (10)$$

In Eq. (10) as $E \to 0$ $T \to 0$.
With little energy probability of getting past barrier is small.

As $E \to \infty$, $T \to 1$. Lots of energy guarentees particle will make it past the delta barrier.