QM5: The Hamiltonian of a quantum system is given by $H = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} + \begin{pmatrix} 0 & -i\beta \\ i\beta & 0 \end{pmatrix}$ where α and β are real constants.

a. Find the energy eigenvalues and eigenfunctions.
b. With $\beta \ll \alpha$ treat the 2nd term as a small perturbation, find the first-order correction to the energy eigenvalues and the corresponding eigenfunctions. Compare with your results in (a).

a)

$$|H - EI| = 0$$

$$\begin{vmatrix} \alpha - E & -i\beta \\ i\beta & \alpha - E \end{vmatrix} = 0$$

$$(\alpha - E)^2 - \beta^2 = 0$$

$$(\alpha - E)^2 = \beta^2$$

$$\alpha - E = \pm\beta$$

$$\boxed{E_\pm = \alpha \pm \beta}$$

E_+ eigenvector

$$H\vec{V}^+ = E_+ \vec{V}^+$$

$$\begin{pmatrix} \alpha & -i\beta \\ i\beta & \alpha \end{pmatrix} \begin{pmatrix} V_1^+ \\ V_2^+ \end{pmatrix} = E_+ \begin{pmatrix} V_1^+ \\ V_2^+ \end{pmatrix}$$

Using first row,

$$\alpha V_1^+ - i\beta V_2^+ = (\alpha + \beta) V_1^+$$

$$-i\cancel{\beta}\, v_2^+ = \cancel{\beta} v_1^+$$

$$\Rightarrow \frac{v_2^+}{v_1^+} = i$$

Therefore, the normalized eigenvector is

$$\boxed{\vec{v}^+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}}$$

E_- eigenvector

$$\begin{pmatrix} \alpha & -i\beta \\ i\beta & \alpha \end{pmatrix} \vec{V}^- = (\alpha - \beta)\, \vec{V}^-$$

Again, using top row,

$$\alpha V_1^- - i\beta V_2^- = (\alpha - \beta) V_1^-$$

$$-i\beta V_2^- = -\beta V_1^-$$

$$\frac{V_2^-}{V_1^-} = -i$$

Then,

$$\boxed{\vec{V}^- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}}$$

b) Now, $$H_0 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$$

We must use degenerate perturbation theory, which tells us to diagonalize the perturbation Hamiltonian.

$$H' = \begin{pmatrix} 0 & -i\beta \\ i\beta & 0 \end{pmatrix}$$

$$0 = \left| H' - E I \right|$$

$$0 = \begin{vmatrix} -E & -i\beta \\ i\beta & -E \end{vmatrix}$$

$$0 = E^2 - \beta^2$$

$$E = \pm \beta$$

Thus, $$\boxed{E_\pm^{(1)} = \pm \beta}$$

as expected.

Now we get the eigenvectors.

$E_+^{(1)} = \beta$

$$H'\vec{v} = \beta \vec{V}$$

$$\begin{pmatrix} 0 & -i\beta \\ i\beta & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \beta \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

$$-i\beta V_2 = \beta V_1$$

$$\frac{V_2}{V_1} = i \Rightarrow \boxed{V_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}}$$

$E_-^{(1)} = \beta$

$$H'\vec{v} = -\beta \vec{V}$$

$$\begin{pmatrix} 0 & -i\beta \\ i\beta & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = -\beta \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

$$-i\beta V_2 = -\beta V_1$$

$$\frac{V_2}{V_1} = -i \qquad \boxed{\vec{V}_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}}$$

Therefore, the energy eigenstates

and eigenfunctions are:

- $E_+ = E_+^{(0)} + E_+^{(1)} = \alpha + \beta$

  $\vec{V}_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$

- $E_- = E_-^{(0)} + E_-^{(1)} = \alpha - \beta$

  $\vec{V}_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$

as expected.