

1. Five men and nine women stand equally spaced around a circle in random order. The probability that every man stands diametrically opposite a woman is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

五名男士和九名女士随机等间距的站成一个圆圈。站在每名男士对径点的人均为女士的概率是 $\frac{m}{n}$ ，其中 m 和 n 是互质的正整数。求 $m + n$ 。

Answer | 答案: 191

2. Positive real numbers $b \neq 1$ and n satisfy the equations

$$\sqrt{\log_b n} = \log_b \sqrt{n} \quad \text{and} \quad b \cdot \log_b n = \log_b(bn).$$

The value of n is $\frac{j}{k}$, where j and k are relatively prime positive integers. Find $j + k$.

正实数 $b \neq 1$ 与 n 满足方程

$$\sqrt{\log_b n} = \log_b \sqrt{n} \quad \text{和} \quad b \cdot \log_b n = \log_b(bn).$$

n 的值是 $\frac{j}{k}$ ，其中 j 和 k 是互质的正整数。求 $j + k$ 。

Answer | 答案: 881

3. A plane contains 40 lines, no 2 of which are parallel. Suppose that there are 3 points where exactly 3 lines intersect, 4 points where exactly 4 lines intersect, 5 points where exactly 5 lines intersect, 6 points where exactly 6 lines intersect, and no points where more than 6 lines intersect. Find the number of points where exactly 2 lines intersect.

平面上有 40 条直线，其中任意两条直线都不平行。假设恰好有 3 条直线经过的交点有 3 个，恰好有 4 条直线经过的交点有 4 个，恰好有 5 条直线经过的交点有 5 个，恰好有 6 条直线经过的交点有 6 个，没有多于 6 条直线相交于同一点。求恰好由 2 条直线相交形成的交点的个数。

Answer | 答案: 607

4. The sum of all positive integers m such that $\frac{13!}{m}$ is a perfect square can be written as $2^a 3^b 5^c 7^d 11^e 13^f$, where $a, b, c, d, e,$ and f are positive integers. Find $a + b + c + d + e + f$.

所有使得 $\frac{13!}{m}$ 是完全平方数的正整数 m 的总和可以写成 $2^a 3^b 5^c 7^d 11^e 13^f$ 的形式, 其中 a, b, c, d, e, f 是正整数. 求 $a + b + c + d + e + f$.

Answer | 答案: 012

5. Let P be a point on the circle circumscribing square $ABCD$ that satisfies $PA \cdot PC = 56$ and $PB \cdot PD = 90$. Find the area of $ABCD$.

设 P 是正方形 $ABCD$ 的外接圆上的一点, 满足 $PA \cdot PC = 56, PB \cdot PD = 90$. 求 $ABCD$ 的面积.

Answer | 答案: 106

6. Alice knows that 3 red cards and 3 black cards will be revealed to her one at a time in random order. Before each card is revealed, Alice must guess its color. If Alice plays optimally, the expected number of cards she will guess correctly is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Alice 知道 3 张红牌和 3 张黑牌将被随机的一张一张地展示给她. 在看到每张牌之前, Alice 需要对它的颜色做出猜测. 如果 Alice 按最佳策略行事, 她猜对颜色的牌数的期望值是 $\frac{m}{n}$, 其中 m 和 n 是互质的正整数. 求 $m + n$.

Answer | 答案: 051

7. Call a positive integer n *extra-distinct* if the remainders when n is divided by 2, 3, 4, 5, and 6 are distinct. Find the number of extra-distinct positive integers less than 1000.

如果 n 除以 2, 3, 4, 5, 6 的余数互不相同, 则称正整数 n 是“特别不同”的. 求小于 1000 的特别不同的正整数的个数.

Answer | 答案: 049

8. Rhombus $ABCD$ has $\angle BAD < 90^\circ$. There is a point P on the incircle of the rhombus such that the distances from P to the lines DA , AB , and BC are 9, 5, and 16, respectively. Find the perimeter of $ABCD$.

在菱形 $ABCD$ 中, $\angle BAD < 90^\circ$. 在菱形的内切圆上有一点 P , 使得从 P 到直线 DA , AB , BC 的距离分别为 9, 5, 16. 求 $ABCD$ 的周长.

Answer | 答案: 125

9. Find the number of cubic polynomials $p(x) = x^3 + ax^2 + bx + c$, where a, b , and c are integers in $\{-20, -19, -18, \dots, 18, 19, 20\}$, such that there is a unique integer $m \neq 2$ with $p(m) = p(2)$.

考虑三次多项式 $p(x) = x^3 + ax^2 + bx + c$, 其中 a, b, c 是 $\{-20, -19, -18, \dots, 18, 19, 20\}$ 中的整数, 并且存在唯一的整数 $m \neq 2$ 满足 $p(m) = p(2)$. 求这样的多项式的个数.

Answer | 答案: 738

10. There exists a unique positive integer a such that the sum

$$U = \sum_{n=1}^{2023} \left\lfloor \frac{n^2 - na}{5} \right\rfloor$$

is an integer strictly between -1000 and 1000 . For that unique a , find $a + U$. Here $\lfloor x \rfloor$ is the greatest integer less than or equal to x .

已知存在唯一的正整数 a 使得总和

$$U = \sum_{n=1}^{2023} \left\lfloor \frac{n^2 - na}{5} \right\rfloor$$

是严格介于 -1000 与 1000 之间的整数. 对于这样唯一的 a , 求 $a + U$. 这里 $\lfloor x \rfloor$ 是小于或等于 x 的最大整数.

Answer | 答案: 944

11. Find the number of subsets of $\{1, 2, 3, \dots, 10\}$ that contain exactly one pair of consecutive integers. Examples of such subsets are $\{1, 2, 5\}$ and $\{1, 3, 6, 7, 10\}$.

求恰好包含一对连续整数的 $\{1, 2, 3, \dots, 10\}$ 的子集数. 这样子集的例子有 $\{1, 2, 5\}$ 和 $\{1, 3, 6, 7, 10\}$.

Answer | 答案: 235

12. Let $\triangle ABC$ be an equilateral triangle with side length 55. Points D , E , and F lie on \overline{BC} , \overline{CA} , and \overline{AB} , respectively, with $BD = 7$, $CE = 30$, and $AF = 40$. Point P inside $\triangle ABC$ has the property that

$$\angle AEP = \angle BFP = \angle CDP.$$

Find $\tan^2(\angle AEP)$.

设 $\triangle ABC$ 是边长为 55 的等边三角形. 点 D , E , F 分别在边 \overline{BC} , \overline{CA} , \overline{AB} 上, 使得 $BD = 7$, $CE = 30$, $AF = 40$. $\triangle ABC$ 内的点 P 具有性质

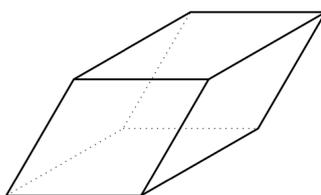
$$\angle AEP = \angle BFP = \angle CDP.$$

求 $\tan^2(\angle AEP)$.

Answer | 答案: 075

13. Each face of two noncongruent parallelepipeds is a rhombus whose diagonals have lengths $\sqrt{21}$ and $\sqrt{31}$. The ratio of the volume of the larger of the two polyhedra to the volume of the smaller is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$. A parallelepiped is a solid with six parallelogram faces such as the one shown below.

两个不全等的平行六面体的每个面都是对角线长度为 $\sqrt{21}$ 与 $\sqrt{31}$ 的菱形. 两个多面体中较大多面体的体积与较小多面体的体积之比为 $\frac{m}{n}$, 其中 m 和 n 是互质的正整数. 求 $m + n$. 平行六面体是指六个面都是平行四边形的立体图形, 一个例子如下图所示.



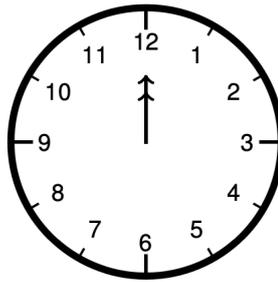
Answer | 答案: 125

14. The following analog clock has two hands that can move independently of each other. Initially, both hands point to the number 12. The clock performs a sequence of hand movements so that on each movement, one of the two hands moves clockwise to the next number on the clock face while the other hand does not move.

Let N be the number of sequences of 144 hand movements such that during the sequence, every possible positioning of the hands appears exactly once, and at the end of the 144 movements, the hands have returned to their initial position. Find the remainder when N is divided by 1000.

下面的模拟时钟有两个可以彼此独立运转的指针。最初，两个指针均指向数字 12。指针移动序列是指时钟的一系列指针移动，每次移动时，两个指针中的一个顺时针移动到钟面上的下一个数字，而另一个则保持不动。

考虑包含 144 次移动的指针移动序列，在整个过程中，两个指针的每个可能位置都恰好出现一次，并且在 144 次移动结束时，指针恰好返回到其初始位置。设 N 是这样的指针移动序列的个数。求 N 除以 1000 的余数。



Answer | 答案: 608

15. Find the greatest prime number $p < 1000$ such that there exists a complex number z satisfying
- the real part of z and the imaginary part of z are both integers,
 - $|z| = \sqrt{p}$, and
 - there exists a triangle whose three side lengths are p , the real part of z^3 , and the imaginary part of z^3 .

求最大的质数 $p < 1000$ ，使得存在复数 z 满足

- z 的实部和 z 的虚部都是整数，
- $|z| = \sqrt{p}$ ，
- 存在一个三角形，其三边长为 p ， z^3 的实部， z^3 的虚部。

Answer | 答案: 349