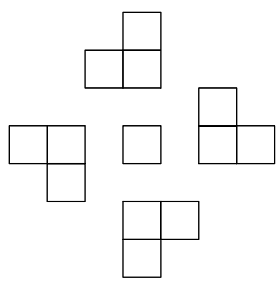
AIME Problems 2013

–	I
–	March 14th
1	The AIME Triathlon consists of a half-mile swim, a 30-mile bicycle, and an eight-mile run. Tom swims, bicycles, and runs at constant rates. He runs five times as fast as he swims, and he bicycles twice as fast as he runs. Tom completes the AIME Triathlon in four and a quarter hours. How many minutes does he spend bicycling?
2	Find the number of five-digit positive integers, n, that satisfy the following conditions: (a) the number n is divisible by 5, (b) the first and last digits of n are equal, and (c) the sum of the digits of n is divisible by 5.
3	Let $ABCD$ be a square, and let E and F be points on $\overline{AB}$ and $\overline{BC}$, respectively. The line through E parallel to $\overline{BC}$ and the line through F parallel to $\overline{AB}$ divide $ABCD$ into two squares and two non square rectangles. The sum of the areas of the two squares is $\frac{9}{10}$ of the area of square $ABCD$. Find $\frac{AE}{EB}+\frac{EB}{AE}$.
4	In the array of 13 squares shown below, 8 squares are colored red, and the remaining 5 squares are colored blue. If one of all possible such colorings is chosen at random, the probability that the chosen colored array appears the same when rotated 90° around the central square is $\frac{1}{n}$, where n is a positive integer. Find n. 
5	The real root of the equation $8x^3-3x^2-3x-1=0$ can be written in the form $\frac{\sqrt[3]{a}+\sqrt[3]{b}+1}{c}$, where a, b, and c are positive integers. Find $a+b+c$.

6 Melinda has three empty boxes and 12 textbooks, three of which are mathematics textbooks. One box will hold any three of her textbooks, one will hold any four of her textbooks, and one will hold any five of her textbooks. If Melinda packs her textbooks into these boxes in random order, the probability that all three mathematics textbooks end up in the same box can be written as $\frac{m}{n}$, where m and n Are relatively prime positive integers. Find $m+n$.

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7 A rectangular box has width 12 inches, length 16 inches, and height $\frac{m}{n}$ inches, where m and n are relatively prime positive integers. Three faces of the box meet at a corner of the box. The center points of those three faces are the vertices of a triangle with an area of 30 square inches. Find $m+n$.

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8 The domain of the function $f(x) = \arcsin(\log_m(nx))$ is a closed interval of length $\frac{1}{2013}$, where m and n are positive integers and $m > 1$. Find the remainder when the smallest possible sum $m+n$ is divided by 1000.

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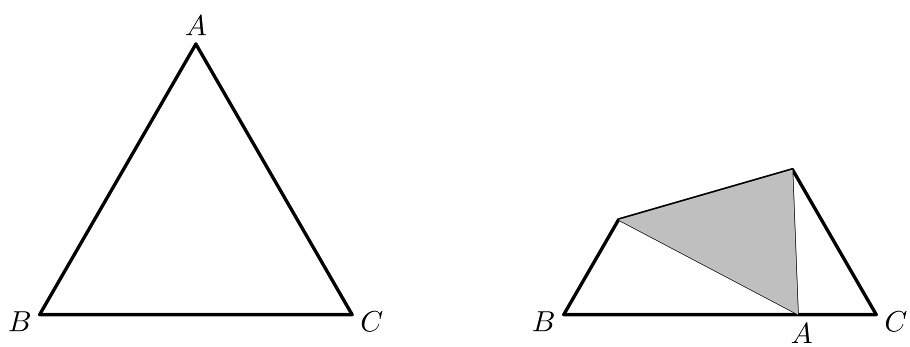
9 A paper equilateral triangle ABC has side length 12. The paper triangle is folded so that vertex A touches a point on side $\overline{BC}$ a distance 9 from point B. The length of the line segment along which the triangle is folded can be written as $\frac{m\sqrt{p}}{n}$, where m, n, and p are positive integers, m and n are relatively prime, and p is not divisible by the square of any prime. Find $m+n+p$.



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10 There are nonzero integers a, b, r, and s such that the complex number $r+si$ is a zero of the polynomial $P(x) = x^3 - ax^2 + bx - 65$. For each possible combination of a and b, let $p_{a,b}$ be the sum of the zeroes of $P(x)$. Find the sum of the $p_{a,b}$'s for all possible combinations of a and b.

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11 Ms. Math's kindergarten class has 16 registered students. The classroom has a very large number, N, of play blocks which satisfies the conditions:

(a) If 16, 15, or 14 students are present, then in each case all the blocks can be distributed in equal numbers to each student, and
(b) There are three integers $0 < x < y < z < 14$ such that when x, y, or z students are present

and the blocks are distributed in equal numbers to each student, there are exactly three blocks left over.

Find the sum of the distinct prime divisors of the least possible value of N satisfying the above conditions.

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12 Let $\triangle PQR$ be a triangle with $\angle P = 75^\circ$ and $\angle Q = 60^\circ$. A regular hexagon $ABCDEF$ with side length 1 is drawn inside $\triangle PQR$ so that side $\overline{AB}$ lies on $\overline{PQ}$, side $\overline{CD}$ lies on $\overline{QR}$, and one of the remaining vertices lies on $\overline{RP}$. There are positive integers a, b, c, and d such that the area of $\triangle PQR$ can be expressed in the form $\frac{a+b\sqrt{c}}{d}$, where a and d are relatively prime and c is not divisible by the square of any prime. Find $a+b+c+d$.

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13 Triangle AB_0C_0 has side lengths $AB_0 = 12$, $B_0C_0 = 17$, and $C_0A = 25$. For each positive integer n, points B_n and C_n are located on $\overline{AB_{n-1}}$ and $\overline{AC_{n-1}}$, respectively, creating three similar triangles $\triangle AB_nC_n \sim \triangle B_{n-1}C_nC_{n-1} \sim \triangle AB_{n-1}C_{n-1}$. The area of the union of all triangles $B_{n-1}C_nB_n$ for $n \geq 1$ can be expressed as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find q.

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14 For $\pi \leq \theta < 2\pi$, let

$$P = \frac{1}{2}\cos\theta - \frac{1}{4}\sin 2\theta - \frac{1}{8}\cos 3\theta + \frac{1}{16}\sin 4\theta + \frac{1}{32}\cos 5\theta - \frac{1}{64}\sin 6\theta - \frac{1}{128}\cos 7\theta + \ldots$$

and

$$Q = 1 - \frac{1}{2}\sin\theta - \frac{1}{4}\cos 2\theta + \frac{1}{8}\sin 3\theta + \frac{1}{16}\cos 4\theta - \frac{1}{32}\sin 5\theta - \frac{1}{64}\cos 6\theta + \frac{1}{128}\sin 7\theta + \ldots$$

so that $\frac{P}{Q} = \frac{2\sqrt{2}}{7}$. Then $\sin\theta = -\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m+n$.

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15 Let N be the number of ordered triples (A, B, C) of integers satisfying the conditions

(a) $0 \leq A < B < C \leq 99$,
(b) there exist integers a, b, and c, and prime p where $0 \leq b < a < c < p$,
(c) p divides $A-a$, $B-b$, and $C-c$, and
(d) each ordered triple (A, B, C) and each ordered triple (b, a, c) form arithmetic sequences.

Find N.

– II

– April 3rd

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1 Suppose that the measurement of time during the day is converted to the metric system so that each day has 10 metric hours, and each metric hour has 100 metric minutes. Digital clocks would then be produced that would read $9{:}99$ just before midnight, $0{:}00$ at midnight, $1{:}25$ at the former $3{:}00$ am, and $7{:}50$ at the former $6{:}00$ pm. After the conversion, a person who wanted to wake up at the equivalent of the former $6{:}36$ am would have to set his new digital alarm clock for A:BC, where A, B, and C are digits. Find 100A $+$ 10B $+$ C.

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2 Positive integers a and b satisfy the condition

$$\log_2(\log_{2^a}(\log_{2^b}(2^{1000}))) = 0.$$

Find the sum of all possible values of $a + b$.

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3 A large candle is 119 centimeters tall. It is designed to burn down more quickly when it is first lit and more slowly as it approaches its bottom. Specifically, the candle takes 10 seconds to burn down the first centimeter from the top, 20 seconds to burn down the second centimeter, and $10k$ seconds to burn down the k-th centimeter. Suppose it takes T seconds for the candle to burn down completely. Then $\frac{T}{2}$ seconds after it is lit, the candle's height in centimeters will be h. Find $10h$.

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4 In the Cartesian plane let $A = (1, 0)$ and $B = \left(2, 2\sqrt{3}\right)$. Equilateral triangle ABC is constructed so that C lies in the first quadrant. Let $P = (x, y)$ be the center of $\triangle ABC$. Then $x \cdot y$ can be written as $\frac{p\sqrt{q}}{r}$, where p and r are relatively prime positive integers and q is an integer that is not divisible by the square of any prime. Find $p + q + r$.

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5 In equilateral $\triangle ABC$ let points D and E trisect $\overline{BC}$. Then $\sin(\angle DAE)$ can be expressed in the form $\frac{a\sqrt{b}}{c}$, where a and c are relatively prime positive integers, and b is an integer that is not divisible by the square of any prime. Find $a + b + c$.

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6 Find the least positive integer N such that the set of 1000 consecutive integers beginning with $1000 \cdot N$ contains no square of an integer.

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7 A group of clerks is assigned the task of sorting 1775 files. Each clerk sorts at a constant rate of 30 files per hour. At the end of the first hour, some of the clerks are reassigned to another task; at the end of the second hour, the same number of the remaining clerks are also reassigned to another task, and a similar reassignment occurs at the end of the third hour. The group finishes the sorting in 3 hours and 10 minutes. Find the number of files sorted during the first one and a half hours of sorting.

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8 A hexagon that is inscribed in a circle has side lengths 22, 22, 20, 22, 22, and 20 in that order. The radius of the circle can be written as $p + \sqrt{q}$, where p and q are positive integers. Find $p + q$.

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9 A 7×1 board is completely covered by $m \times 1$ tiles without overlap; each tile may cover any number of consecutive squares, and each tile lies completely on the board. Each tile is either red, blue, or green. Let N be the number of tilings of the 7×1 board in which all three colors are used at least once. For example, a 1×1 red tile followed by a 2×1 green tile, a 1×1 green tile, a 2×1 blue tile, and a 1×1 green tile is a valid tiling. Note that if the 2×1 blue tile is replaced by two 1×1 blue tiles, this results in a different tiling. Find the remainder when N is divided by 1000.

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10 Given a circle of radius $\sqrt{13}$, let A be a point at a distance $4 + \sqrt{13}$ from the center O of the circle. Let B be the point on the circle nearest to point A. A line passing through the point A intersects the circle at points K and L. The maximum possible area for $\triangle BKL$ can be written in the form $\frac{a-b\sqrt{c}}{d}$, where a, b, c, and d are positive integers, a and d are relatively prime, and c is not divisible by the square of any prime. Find $a+b+c+d$.

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11 Let $A = \{1,2,3,4,5,6,7\}$ and let N be the number of functions f from set A to set A such that $f(f(x))$ is a constant function. Find the remainder when N is divided by 1000.

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12 Let S be the set of all polynomials of the form $z^3 + az^2 + bz + c$, where a, b, and c are integers. Find the number of polynomials in S such that each of its roots z satisfies either $|z| = 20$ or $|z| = 13$.

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13 In $\triangle ABC$, $AC = BC$, and point D is on $\overline{BC}$ so that $CD = 3 \cdot BD$. Let E be the midpoint of $\overline{AD}$. Given that $CE = \sqrt{7}$ and $BE = 3$, the area of $\triangle ABC$ can be expressed in the form $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m+n$.

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14 For positive integers n and k, let $f(n,k)$ be the remainder when n is divided by k, and for $n > 1$ let $F(n) = \max\limits_{1 \le k \le \frac{n}{2}} f(n,k)$. Find the remainder when $\sum\limits_{n=20}^{100} F(n)$ is divided by 1000.

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15 Let A, B, C be angles of an acute triangle with

$$\cos^2 A + \cos^2 B + 2\sin A \sin B \cos C = \frac{15}{8} \text{ and}$$
$$\cos^2 B + \cos^2 C + 2\sin B \sin C \cos A = \frac{14}{9}.$$

There are positive integers p, q, r, and s for which

$$\cos^2 C + \cos^2 A + 2\sin C \sin A \cos B = \frac{p - q\sqrt{r}}{s},$$

where $p+q$ and s are relatively prime and r is not divisible by the square of any prime. Find $p+q+r+s$.

2013 AIME I Answer Key

1. 150
2. 200
3. 018
4. 429
5. 098
6. 047
7. 041
8. 371
9. 113
10. 080
11. 148
12. 021
13. 961
14. 036
15. 272

2013 AIME II Answer Key

1. 275
2. 881
3. 350
4. 040
5. 020
6. 282
7. 945
8. 272
9. 106
10. 146
11. 399
12. 540
13. 010
14. 512
15. 222