MAA AMC

American Mathematics Competitions

MAA American Mathematics Competitions

40th Annual

AIME I

American Invitational Mathematics Examination I

Tuesday, February 8, 2022

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOUR COMPETITION MANAGER TELLS YOU TO BEGIN.
2. This is a 15-question competition. All answers are integers ranging from 000 to 999, inclusive.
3. Mark your answer to each problem on the answer sheet with a #2 pencil. Check blackened answers for accuracy and erase errors completely. Only answers that are properly marked on the answer sheet will be scored.
4. SCORING: You will receive 1 point for each correct answer, 0 points for each problem left unanswered, and 0 points for each incorrect answer.
5. Only blank scratch paper, rulers, compasses, and erasers are allowed as aids. Prohibited materials include calculators, smartwatches, phones, computing devices, protractors, and graph paper.
6. Figures are not necessarily drawn to scale.
7. Before beginning the competition, your competition manager will ask you to record your name on the answer sheet.
8. You will have 3 hours to complete the competition once your competition manager tells you to begin.
9. When you finish the competition, sign your name in the space provided on the answer sheet.

The MAA AMC Office reserves the right to disqualify scores from a school if it determines that the rules or the required security procedures were not followed.

The publication, reproduction, or communication of the problems or solutions of this competition during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via phone, email, or digital media of any type during this period is a violation of the competition rules.

A combination of your AIME score and your AMC 10/12 score is used to determine eligibility for participation in the USA (Junior) Mathematical Olympiad.

Problem 1:

Quadratic polynomials $P(x)$ and $Q(x)$ have leading coefficients of 2 and -2, respectively. The graphs of both polynomials pass through the two points $(16, 54)$ and $(20, 53)$. Find $P(0) + Q(0)$.

Problem 2:

Find the three-digit positive integer $\underline{a}\,\underline{b}\,\underline{c}$ whose representation in base nine is $\underline{b}\,\underline{c}\,\underline{a}_{\,\text{nine}}$, where a, b, and c are (not necessarily distinct) digits.

Problem 3:

In isosceles trapezoid $ABCD$, parallel bases $\overline{AB}$ and $\overline{CD}$ have lengths 500 and 650, respectively, and $AD = BC = 333$. The angle bisectors of $\angle A$ and $\angle D$ meet at P, and the angle bisectors of $\angle B$ and $\angle C$ meet at Q. Find PQ.

Problem 4:

Let $w = \frac{\sqrt{3}+i}{2}$ and $z = \frac{-1+i\sqrt{3}}{2}$, where $i = \sqrt{-1}$. Find the number of ordered pairs (r, s) of positive integers not exceeding 100 that satisfy the equation $i \cdot w^r = z^s$.

Problem 5:

A straight river that is 264 meters wide flows from west to east at a rate of 14 meters per minute. Melanie and Sherry sit on the south bank of the river with Melanie a distance of D meters downstream from Sherry. Relative to the water, Melanie swims at 80 meters per minute, and Sherry swims at 60 meters per minute. At the same time, Melanie and Sherry begin swimming in straight lines to a point on the north bank of the river that is equidistant from their starting positions. The two women arrive at this point simultaneously. Find D.

Problem 6:

Find the number of ordered pairs of integers (a, b) such that the sequence

$$3, 4, 5, a, b, 30, 40, 50$$

is strictly increasing and no set of four (not necessarily consecutive) terms forms an arithmetic progression.

Problem 7:

Let a, b, c, d, e, f, g, h, i be distinct integers from 1 to 9. The minimum possible positive value of

$$\frac{a \cdot b \cdot c - d \cdot e \cdot f}{g \cdot h \cdot i}$$

can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 8:

Equilateral triangle $\triangle ABC$ is inscribed in circle ω with radius 18. Circle ω_A is tangent to sides $\overline{AB}$ and $\overline{AC}$ and is internally tangent to ω. Circles ω_B and ω_C are defined analogously. Circles ω_A, ω_B, and ω_C meet in six points—two points for each pair of circles. The three intersection points closest to the vertices of $\triangle ABC$ are the vertices of a large equilateral triangle in the interior of $\triangle ABC$, and the other three intersection points are the vertices of a smaller equilateral triangle in the interior of $\triangle ABC$. The side length of the smaller equilateral triangle can be written as $\sqrt{a} - \sqrt{b}$, where a and b are positive integers. Find $a + b$.

Problem 9:

Ellina has twelve blocks, two each of red (**R**), blue (**B**), yellow (**Y**), green (**G**), orange (**O**), and purple (**P**). Call an arrangement of blocks *even* if there is an even number of blocks between each pair of blocks of the same color. For example, the arrangement

R B B Y G G Y R O P P O

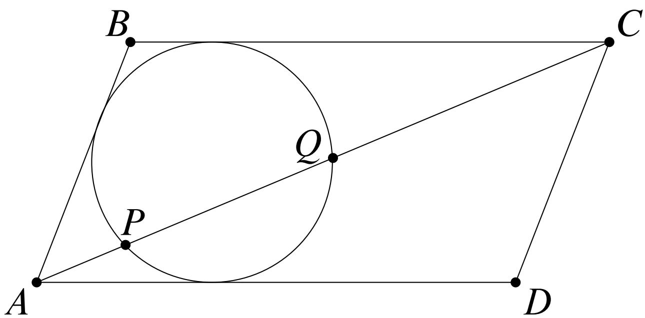
is even. Ellina arranges her blocks in a row in random order. The probability that her arrangement is even is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 10:

Three spheres with radii 11, 13, and 19 are mutually externally tangent. A plane intersects the spheres in three congruent circles centered at A, B, and C, respectively, and the centers of the spheres all lie on the same side of this plane. Suppose that $AB^2 = 560$. Find AC^2.

Problem 11:

Let $ABCD$ be a parallelogram with $\angle BAD < 90^\circ$. A circle tangent to sides $\overline{DA}$, $\overline{AB}$, and $\overline{BC}$ intersects diagonal $\overline{AC}$ at points P and Q with $AP < AQ$, as shown. Suppose that $AP = 3$, $PQ = 9$, and $QC = 16$. Then the area of $ABCD$ can be expressed in the form $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m + n$.



Problem 12:

For any finite set X, let $|X|$ denote the number of elements in X. Define

$$S_n = \sum |A \cap B|,$$

where the sum is taken over all ordered pairs (A, B) such that A and B are subsets of $\{1, 2, 3, \ldots, n\}$ with $|A| = |B|$. For example, $S_2 = 4$ because the sum is taken over the pairs of subsets

$$(A, B) \in \big\{(\emptyset, \emptyset), (\{1\}, \{1\}), (\{1\}, \{2\}), (\{2\}, \{1\}), (\{2\}, \{2\}), (\{1, 2\}, \{1, 2\})\big\},$$

giving $S_2 = 0 + 1 + 0 + 0 + 1 + 2 = 4$. Let $\frac{S_{2022}}{S_{2021}} = \frac{p}{q}$, where p and q are relatively prime positive integers. Find the remainder when $p + q$ is divided by 1000.

Problem 13:

Let S be the set of all rational numbers that can be expressed as a repeating decimal in the form $0.\overline{abcd}$, where at least one of the digits a, b, c, or d is nonzero. Let N be the number of distinct numerators obtained when numbers in S are written as fractions in lowest terms. For example, both 4 and 410 are counted among the distinct numerators for numbers in S because $0.\overline{3636} = \frac{4}{11}$ and $0.\overline{1230} = \frac{410}{3333}$. Find the remainder when N is divided by 1000.

Problem 14:

Given $\triangle ABC$ and a point P on one of its sides, call line ℓ the *splitting line* of $\triangle ABC$ through P if ℓ passes through P and divides $\triangle ABC$ into two polygons of equal perimeter. Let $\triangle ABC$ be a triangle where $BC = 219$ and AB and AC are positive integers. Let M and N be the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively, and suppose that the splitting lines of $\triangle ABC$ through M and N intersect at 30°. Find the perimeter of $\triangle ABC$.

Problem 15:

Let x, y, and z be positive real numbers satisfying the system of equations

$$\begin{aligned}\sqrt{2x - xy} + \sqrt{2y - xy} &= 1\\ \sqrt{2y - yz} + \sqrt{2z - yz} &= \sqrt{2}\\ \sqrt{2z - zx} + \sqrt{2x - zx} &= \sqrt{3}.\end{aligned}$$

Then $\left[(1-x)(1-y)(1-z)\right]^2$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

The problems and solutions for the American Invitational Mathematics Exams are selected and edited by the AIME Editorial Board of the MAA, with Co-Editors-in-Chief Jonathan Kane and Sergey Levin. The problems appearing on this competition were authored by Chris Jeuell, David Altizio, David Wells, Elgin Johnston, Jerrold Grossman, Jonathan Kane, Matyas A. Sustik, Mehtaab Sawhney, Sergey Levin, and Zuming Feng. We thank them all for their contributions.

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Scores and official competition solutions will be sent to your competition manager who can share that information with you.

For more information about the MAA American Mathematics Competitions program, please visit maa.org/amc.

Questions and comments about this competition should be sent to:

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Jonathan Kane and Sergey Levin, co-Editors-in-Chief

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