MAA AMC
American Mathematics Competitions

MAA American Mathematics Competitions

40th Annual

AIME II

American Invitational Mathematics Examination II

Wednesday, February 16, 2022

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOUR COMPETITION MANAGER TELLS YOU TO BEGIN.
2. This is a 15-question competition. All answers are integers ranging from 000 to 999, inclusive.
3. Mark your answer to each problem on the answer sheet with a #2 pencil. Check blackened answers for accuracy and erase errors completely. Only answers that are properly marked on the answer sheet will be scored.
4. SCORING: You will receive 1 point for each correct answer, 0 points for each problem left unanswered, and 0 points for each incorrect answer.
5. Only blank scratch paper, rulers, compasses, and erasers are allowed as aids. Prohibited materials include calculators, smartwatches, phones, computing devices, protractors, and graph paper.
6. Figures are not necessarily drawn to scale.
7. Before beginning the competition, your competition manager will ask you to record your name on the answer sheet.
8. You will have 3 hours to complete the competition once your competition manager tells you to begin.
9. When you finish the competition, sign your name in the space provided on the answer sheet.

The MAA AMC Office reserves the right to disqualify scores from a school if it determines that the rules or the required security procedures were not followed.

The publication, reproduction, or communication of the problems or solutions of this competition during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via phone, email, or digital media of any type during this period is a violation of the competition rules.

A combination of your AIME score and your AMC 10/12 score is used to determine eligibility for participation in the USA (Junior) Mathematical Olympiad.

Problem 1:

Adults made up $\frac{5}{12}$ of the crowd of people at a concert. After a bus carrying 50 more people arrived, adults made up $\frac{11}{25}$ of the people at the concert. Find the minimum number of adults who could have been at the concert after the bus arrived.

Problem 2:

Azar, Carl, Jon, and Sergey are the four players left in a singles tennis tournament. They are randomly assigned opponents in the semifinal matches, and the winners of those matches play each other in the final match to determine the winner of the tournament. When Azar plays Carl, Azar will win the match with probability $\frac{2}{3}$. When either Azar or Carl plays either Jon or Sergey, Azar or Carl will win the match with probability $\frac{3}{4}$. Assume that outcomes of different matches are independent. The probability that Carl will win the tournament is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

Problem 3:

A right square pyramid with volume 54 has a base with side length 6. The five vertices of the pyramid all lie on a sphere with radius $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 4:

There is a positive real number x not equal to either $\frac{1}{20}$ or $\frac{1}{2}$ such that

$$\log_{20x}(22x) = \log_{2x}(202x).$$

The value $\log_{20x}(22x)$ can be written as $\log_{10}\left(\frac{m}{n}\right)$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 5:

Twenty distinct points are marked on a circle and labeled 1 through 20 in clockwise order. A line segment is drawn between every pair of points whose labels differ by a prime number. Find the number of triangles whose sides are three of these line segments and whose vertices are three distinct points from among the original 20 points.

Problem 6:

Let $x_1 \leq x_2 \leq \cdots \leq x_{100}$ be real numbers such that $|x_1| + |x_2| + \cdots + |x_{100}| = 1$ and $x_1 + x_2 + \cdots + x_{100} = 0$. Among all such 100-tuples of numbers, the greatest value that $x_{76} - x_{16}$ can achieve is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 7:

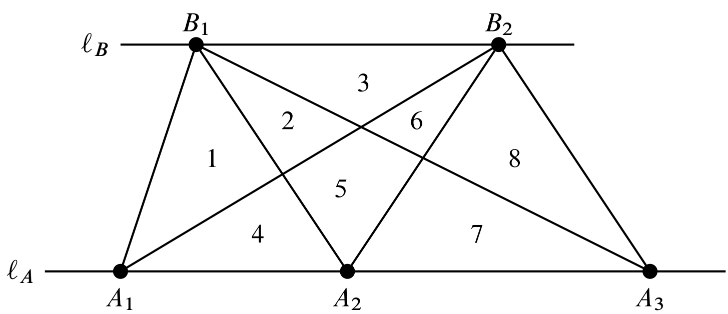
A circle with radius 6 is externally tangent to a circle with radius 24. Find the area of the triangular region bounded by the three common tangent lines of these two circles.

Problem 8:

Find the number of positive integers $n \le 600$ whose value can be uniquely determined when the values of $\left\lfloor \frac{n}{4} \right\rfloor$, $\left\lfloor \frac{n}{5} \right\rfloor$, and $\left\lfloor \frac{n}{6} \right\rfloor$ are given, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to the real number x.

Problem 9:

Let ℓ_A and ℓ_B be two distinct parallel lines. For positive integers m and n, distinct points $A_1, A_2, A_3, \ldots, A_m$ lie on ℓ_A, and distinct points $B_1, B_2, B_3, \ldots, B_n$ lie on ℓ_B. Additionally, when segments $\overline{A_i B_j}$ are drawn for all $i = 1, 2, 3, \ldots, m$ and $j = 1, 2, 3, \ldots, n$, no point strictly between ℓ_A and ℓ_B lies on more than two of the segments. Find the number of bounded regions into which this figure divides the plane when $m = 7$ and $n = 5$. The figure shows that there are 8 regions when $m = 3$ and $n = 2$.



Problem 10:

Find the remainder when

$$\binom{\binom{3}{2}}{2} + \binom{\binom{4}{2}}{2} + \cdots + \binom{\binom{40}{2}}{2}$$

is divided by 1000.

Problem 11:

Let $ABCD$ be a convex quadrilateral with $AB = 2$, $AD = 7$, and $CD = 3$ such that the bisectors of acute angles $\angle DAB$ and $\angle ADC$ intersect at the midpoint of $\overline{BC}$. Find the square of the area of $ABCD$.

Problem 12:

Let a, b, x, and y be real numbers with $a > 4$ and $b > 1$ such that

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - 16} = \frac{(x-20)^2}{b^2 - 1} + \frac{(y-11)^2}{b^2} = 1.$$

Find the least possible value for $a + b$.

Problem 13:

There is a polynomial $P(x)$ with integer coefficients such that

$$P(x) = \frac{(x^{2310} - 1)^6}{(x^{105} - 1)(x^{70} - 1)(x^{42} - 1)(x^{30} - 1)}$$

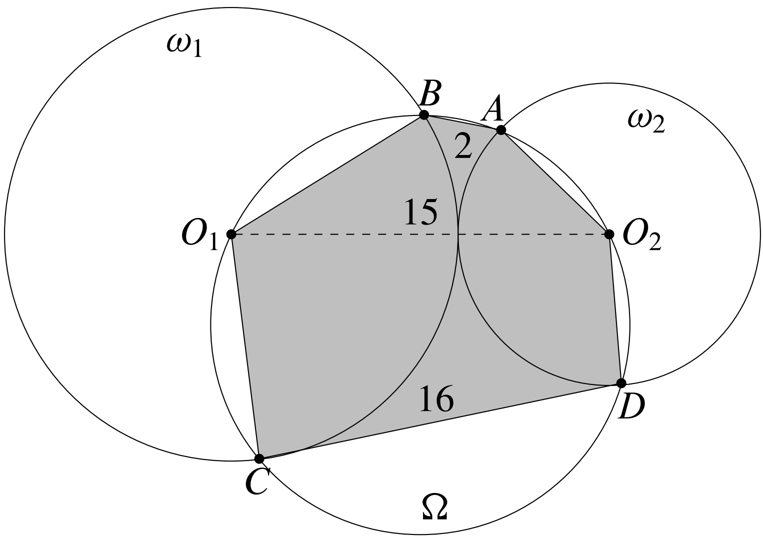
holds for every $0 < x < 1$. Find the coefficient of x^{2022} in $P(x)$.

Problem 14:

For positive integers a, b, and c with $a < b < c$, consider collections of postage stamps in denominations a, b, and c cents that contain at least one stamp of each denomination. If there exists such a collection that contains sub-collections worth every whole number of cents up to 1000 cents, let $f(a, b, c)$ be the minimum number of stamps in such a collection. Find the sum of the three least values of c such that $f(a, b, c) = 97$ for some choice of a and b.

Problem 15:

Two externally tangent circles ω_1 and ω_2 have centers O_1 and O_2, respectively. A third circle Ω passing through O_1 and O_2 intersects ω_1 at B and C and ω_2 at A and D, as shown. Suppose that $AB = 2$, $O_1O_2 = 15$, $CD = 16$, and ABO_1CDO_2 is a convex hexagon. Find the area of this hexagon.



The problems and solutions for the American Invitational Mathematics Exams are selected and edited by the AIME Editorial Board of the MAA, with Co-Editors-in-Chief Jonathan Kane and Sergey Levin. The problems appearing on this competition were authored by Chris Jeuell, David Altizio, David Wells, Evan Chen, Ivan Borsenco, Jerrold Grossman, Jonathan Kane, Michael Tang, and Zachary Franco. We thank them all for their contributions.

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Scores and official competition solutions will be sent to your competition manager who can share that information with you.

For more information about the MAA American Mathematics Competitions program, please visit maa.org/amc.

Questions and comments about this competition should be sent to:

amcinfo@maa.org

or

MAA American Mathematics Competitions
P.O. Box 471
Annapolis Junction, MD 20701

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Jonathan Kane and Sergey Levin, co-Editors-in-Chief

MAA Partner Organizations

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