AIME Problems 2011

–	I
–	March 16th
1	Jar A contains four liters of a solution that is 45% acid. Jar B contains five liters of a solution that is 48% acid. Jar C contains one liter of a solution that is $k\%$ acid. From jar C, $\frac{m}{n}$ liters of the solution is added to jar A, and the remainder of the solution in jar C is added to jar B. At the end, both jar A and jar B contain solutions that are 50% acid. Given that m and n are relatively prime positive integers, find $k+m+n$.
2	In rectangle $ABCD$, $AB = 12$ and $BC = 10$. Points E and F lie inside rectangle $ABCD$ so that $BE = 9$, $DF = 8$, $\overline{BE} \parallel \overline{DF}$, $\overline{EF} \parallel \overline{AB}$, and line BE intersects segment $\overline{AD}$. The length EF can be expressed in the form $m\sqrt{n} - p$, where $m, n,$ and p are positive integers and n is not divisible by the square of any prime. Find $m+n+p$.
3	Let L be the line with slope $\frac{5}{12}$ that contains the point $A = (24, -1)$, and let M be the line perpendicular to line L that contains the point $B = (5, 6)$. The original coordinate axes are erased, and line L is made the x-axis, and line M the y-axis. In the new coordinate system, point A is on the positive x-axis, and point B is on the positive y-axis. The point P with coordinates $(-14, 27)$ in the original system has coordinates (α, β) in the new coordinate system. Find $\alpha + \beta$.
4	In triangle ABC, $AB = 125$, $AC = 117$, and $BC = 120$. The angle bisector of angle A intersects $\overline{BC}$ at point L, and the angle bisector of angle B intersects $\overline{AC}$ at point K. Let M and N be the feet of the perpendiculars from C to $\overline{BK}$ and $\overline{AL}$, respectively. Find MN.
5	The vertices of a regular nonagon (9-sided polygon) are to be labeled with the digits 1 through 9 in such a way that the sum of the numbers on every three consecutive vertices is a multiple of 3. Two acceptable arrangements are considered to be indistinguishable if one can be obtained from the other by rotating the nonagon in the plane. Find the number of distinguishable acceptable arrangements.
6	Suppose that a parabola has vertex $\left(\frac{1}{4}, -\frac{9}{8}\right)$, and equation $y = ax^2 + bx + c$, where $a > 0$ and $a+b+c$ is an integer. The minimum possible value of a can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.
7	Find the number of positive integers m for which there exist nonnegative integers $x_0, x_1, \ldots, x_{2011}$

such that

$$m^{x_0} = \sum_{k=1}^{2011} m^{x_k}.$$

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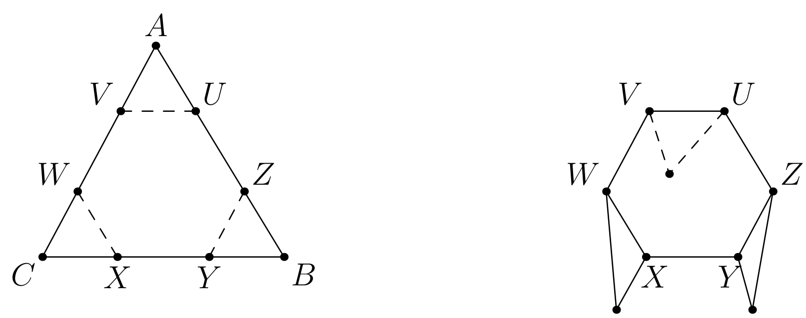
8 In triangle ABC, $BC = 23$, $CA = 27$, and $AB = 30$. Points V and W are on $\overline{AC}$ with V on $\overline{AW}$, points X and Y are on $\overline{BC}$ with X on $\overline{CY}$, and points Z and U are on $\overline{AB}$ with Z on $\overline{BU}$. In addition, the points are positioned so that $\overline{UV} \parallel \overline{BC}$, $\overline{WX} \parallel \overline{AB}$, and $\overline{YZ} \parallel \overline{CA}$. Right angle folds are then made along $\overline{UV}$, $\overline{WX}$, and $\overline{YZ}$. The resulting figure is placed on a level floor to make a table with triangular legs. Let h be the maximum possible height of a table constructed from triangle ABC whose top is parallel to the floor. Then h can be written in the form $\frac{k\sqrt{m}}{n}$, where k and n are relatively prime positive integers and m is a positive integer that is not divisible by the square of any prime. Find $k + m + n$.



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9 Suppose x is in the interval $[0, \pi/2]$ and $\log_{24 \sin x}(24 \cos x) = \frac{3}{2}$.
Find $24 \cot^2 x$.

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10 The probability that a set of three distinct vertices chosen at random from among the vertices of a regular n-gon determine an obtuse triangle is $\frac{93}{125}$. Find the sum of all possible values of n.

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11 Let R be the set of all possible remainders when a number of the form 2^n, n a nonnegative integer, is divided by 1000. Let S be the sum of all elements in R. Find the remainder when S is divided by 1000.

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12 Six men and some number of women stand in a line in random order. Let p be the probability that a group of at least four men stand together in the line, given that every man stands next to at least one other man. Find the least number of women in the line such that p does not exceed 1 percent.

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13 A cube with side length 10 is suspended above a plane. The vertex closest to the plane is labelled A. The three vertices adjacent to vertex A are at heights 10, 11, and 12 above the plane. The distance from vertex A to the plane can be expressed as $\frac{r-\sqrt{s}}{t}$, where r, s, and t are positive integers, and $r+s+t<1000$. Find $r+s+t$.

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14 Let $A_1A_2A_3A_4A_5A_6A_7A_8$ be a regular octagon. Let M_1, M_3, M_5, and M_7 be the midpoints of sides $\overline{A_1A_2}$, $\overline{A_3A_4}$, $\overline{A_5A_6}$, and $\overline{A_7A_8}$, respectively. For $i=1,3,5,7$, ray R_i is constructed from M_i towards the interior of the octagon such that $R_1 \perp R_3$, $R_3 \perp R_5$, $R_5 \perp R_7$, and $R_7 \perp R_1$. Pairs of rays R_1 and R_3, R_3 and R_5, R_5 and R_7, and R_7 and R_1 meet at B_1, B_3, B_5, B_7 respectively. If $B_1B_3 = A_1A_2$, then $\cos 2\angle A_3M_3B_1$ can be written in the form $m-\sqrt{n}$, where m and n are positive integers. Find $m+n$.

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15 For some integer m, the polynomial $x^3-2011x+m$ has the three integer roots a, b, and c. Find $|a|+|b|+|c|$.

– II

– March 31st

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1 Gary purchased a large beverage, but drank only m/n of this beverage, where m and n are relatively prime positive integers. If Gary had purchased only half as much and drunk twice as much, he would have wasted only $\frac{2}{9}$ as much beverage. Find $m+n$.

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2 On square $ABCD$, point E lies on side $\overline{AD}$ and point F lies on side $\overline{BC}$, so that $BE=EF=FD=30$. Find the area of square $ABCD$.

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3 The degree measures of the angles of a convex 18-sided polygon form an increasing arithmetic sequence with integer values. Find the degree measure of the smallest angle.

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4 In triangle ABC, $AB=\frac{20}{11}AC$. The angle bisector of $\angle A$ intersects BC at point D, and point M is the midpoint of AD. Let P be the point of the intersection of AC and BM. The ratio of CP to PA can be expressed in the form $\dfrac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

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5 The sum of the first 2011 terms of a geometric series is 200. The sum of the first 4022 terms of the same series is 380. Find the sum of the first 6033 terms of the series.

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6 Define an ordered quadruple of integers (a,b,c,d) as interesting if $1 \le a<b<c<d \le 10$, and $a+d>b+c$. How many ordered quadruples are there?

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7 Ed has five identical green marbles and a large supply of identical red marbles. He arranges the green marbles and some of the red marbles in a row and finds that the number of marbles whose right hand neighbor is the same color as themselves equals the number of marbles whose right hand neighbor is the other color. An example of such an arrangement is GGRRRGGRG. Let m be the maximum number of red marbles for which Ed can make such an arrangement, and let N be the number of ways in which Ed can arrange the $m+5$ marbles to satisfy the requirement. Find the remainder when N is divided by 1000.

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8 Let $z_1, z_2, z_3, \ldots, z_{12}$ be the 12 zeroes of the polynomial $z^{12} - 2^{36}$. For each j, let w_j be one of z_j or iz_j. Then the maximum possible value of the real part of $\sum_{j=1}^{12} w_j$ can be written as $m + \sqrt{n}$ where m and n are positive integers. Find $m+n$.

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9 Let $x_1, x_2, \ldots, x_6$ be nonnegative real numbers such that $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$, and $x_1x_3x_5 + x_2x_4x_6 \geq \frac{1}{540}$. Let p and q be positive relatively prime integers such that $\frac{p}{q}$ is the maximum possible value of $x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 + x_4x_5x_6 + x_5x_6x_1 + x_6x_1x_2$. Find $p+q$.

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10 A circle with center O has radius 25. Chord $\overline{AB}$ of length 30 and chord $\overline{CD}$ of length 14 intersect at point P. The distance between the midpoints of the two chords is 12. The quantity OP^2 can be represented as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find the remainder where $m+n$ is divided by 1000.

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11 Let M_n be the $n \times n$ matrix with entries as follows: for $1 \leq i \leq n$, $m_{i,i} = 10$; for $1 \leq i \leq n-1$, $m_{i+1,i} = m_{i,i+1} = 3$; all other entries in M_n are zero. Let D_n be the determinant of matrix M_n. Then $\sum_{n=1}^{\infty} \frac{1}{8D_n + 1}$ can be represented as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

Note: The determinant of the 1×1 matrix $[a]$ is a, and the determinant of the 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$; for $n \geq 2$, the determinant of an $n \times n$ matrix with first row or first column $a_1\ a_2\ a_3 \ldots\ a_n$ is equal to $a_1C_1 - a_2C_2 + a_3C_3 - \cdots + (-1)^{n+1}a_nC_n$, where C_i is the determinant of the $(n-1) \times (n-1)$ matrix found by eliminating the row and column containing a_i.

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12 Nine delegates, three each from three different countries, randomly select chairs at a round table that seats nine people. Let the probability that each delegate sits next to at least one delegate from another country be $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

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13 Point P lies on the diagonal AC of square $ABCD$ with $AP > CP$. Let O_1 and O_2 be the circumcenters of triangles ABP and CDP respectively. Given that $AB = 12$ and $\angle O_1PO_2 = 120°$, then $AP = \sqrt{a} + \sqrt{b}$ where a and b are positive integers. Find $a+b$.

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14 There are N permutations $(a_1, a_2, \ldots, a_{30})$ of $1, 2, \ldots, 30$ such that for $m \in \{2, 3, 5\}$, m divides $a_{n+m} - a_n$ for all integers n with $1 \leq n < n+m \leq 30$. Find the remainder when N is divided by 1000.

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15 Let $P(x) = x^2 - 3x - 9$. A real number x is chosen at random from the interval $5 \le x \le 15$. The probability that $\lfloor \sqrt{P(x)} \rfloor = \sqrt{P(\lfloor x \rfloor)}$ is equal to $\frac{\sqrt{a} + \sqrt{b} + \sqrt{c} - d}{e}$, where a, b, c, d and e are positive integers and none of $a, b,$ or c is divisible by the square of a prime. Find $a+b+c+d+e$.

2011 AIME I Answer Key

Return to 2011 AIME I (2011 AIME I Problems)

1. 085
2. 036
3. 031
4. 056
5. 144
6. 011
7. 016
8. 318
9. 192
10. 503
11. 007
12. 594
13. 330
14. 037
15. 098

2011 AIME II Answer Key

Return to 2011 AIME II (2011 AIME II Problems)

1. 037
2. 810
3. 143
4. 051
5. 542
6. 080
7. 003
8. 784
9. 559
10. 057
11. 073
12. 097
13. 096
14. 440
15. 850