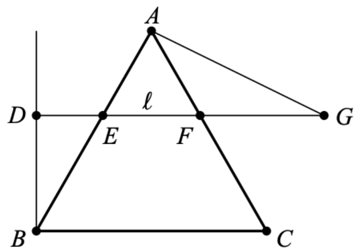
2021 AIME Ⅱ Problems

1. Find the arithmetic mean of all the three-digit palindromes. (Recall that a palindrome is a number that reads the same forward and backward, such as 777 or 383.)

   求所有三位回文数的算术平均值。(注意，一个数如果从前往后读和从后往前读是一样的，则称为回文数，例如 777 和 383。)

2. Equilateral triangle ABC has side length 840. Point D lies on the same side of line BC as A such that $\overline{BD} \perp \overline{BC}$. The line ℓ through D parallel to line BC intersects sides $\overline{AB}$ and $\overline{AC}$ at points E and F, respectively. Point G lies on ℓ such that F is between E and G, $\triangle AFG$ is isosceles, and the ratio of the area of $\triangle AFG$ to the area of $\triangle BED$ is $8:9$. Find AF.

   等边三角形 ABC 的边长是 840。点 D 与 A 位于直线 BC 的同一侧，使得 $\overline{BD} \perp \overline{BC}$。经过 D 的直线 ℓ 与直线 BC 平行，与边 $\overline{AB}$ 和 $\overline{AC}$ 分别相交于点 E 和 F。ℓ 上的点 G，使得 F 在 E 和 G 之间，$\triangle AFG$ 是等腰三角形，并且 $\triangle AFG$ 与 $\triangle BED$ 的面积之比是 $8:9$。求 AF 的长度。



3. Find the number of permutations x_1, x_2, x_3, x_4, x_5 of numbers $1, 2, 3, 4, 5$ such that the sum of five products

   $$x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 + x_4x_5x_1 + x_5x_1x_2$$

   is divisible by 3.

   考虑数 1，2，3，4，5 的排列 x_1，x_2，x_3，x_4，x_5，使得五个乘积之和

   $$x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 + x_4x_5x_1 + x_5x_1x_2$$

   能够被 3 整除。求所有这样的排列的个数。

4. There are real numbers a, b, c, and d such that -20 is a root of $x^3 + ax + b$ and -21 is a root of $x^3 + cx^2 + d$. These two polynomials share a complex root $m + \sqrt{n} \cdot i$, where m and n are positive integers and $i = \sqrt{-1}$. Find $m + n$.

   存在实数 a，b，c，d，使得 -20 是 $x^3 + ax + b$ 的根，而 -21 是 $x^3 + cx^2 + d$ 的根。这两个多项式有一个公共的复数根 $m + \sqrt{n} \cdot i$，其中 m 和 n 是正整数，并且 $i = \sqrt{-1}$。求 $m + n$。

5. For positive real numbers s, let $\mathcal{T}(s)$ denote the set of all obtuse triangles that have area s and two sides with lengths 4 and 10. The set of all s for which $\mathcal{T}(s)$ is nonempty, but all triangles in $\mathcal{T}(s)$ are congruent, is an interval $[a, b)$. Find $a^2 + b^2$.

对于正实数 s，设 $\mathcal{T}(s)$ 表示所有面积为 s，且两边长分别为 4 和 10 的钝角三角形组成的集合。由满足 $\mathcal{T}(s)$ 不是空集，而 $\mathcal{T}(s)$ 中所有三角形都全等的 s 组成的集合是区间 $[a，b)$ 。求 $a^2 + b^2$。

6. For any finite set S, let $|S|$ denote the number of elements in S. Find the number of ordered pairs (A, B) such that A and B are (not necessarily distinct) subsets of {1, 2, 3, 4, 5} that satisfy

$$|A| \cdot |B| = |A \cap B| \cdot |A \cup B|.$$

对于任何有限集合 S，令 $|S|$ 表示 S 中的元素个数。求满足下述条件的有序对 $(A，B)$ 的个数：A 和 B（未必不同）是 $\{1，2，3，4，5\}$ 的子集，并且

$$|A| \cdot |B| = |A \cap B| \cdot |A \cup B|。$$

7. Let $a, b, c,$ and d be real numbers that satisfy the system of equations

$$\begin{aligned} a + b &= -3 \\ ab + bc + ca &= -4 \\ abc + bcd + cda + dab &= 14 \\ abcd &= 30. \end{aligned}$$

There exist relatively prime positive integers m and n such that

$$a^2 + b^2 + c^2 + d^2 = \frac{m}{n}.$$

Find $m + n$.

设实数 $a，b，c，d$ 满足方程组

$$\begin{aligned} a + b &= -3 \\ ab + bc + ca &= -4 \\ abc + bcd + cda + dab &= 14 \\ abcd &= 30。 \end{aligned}$$

存在互质的正整数 m 和 n，使得

$$a^2 + b^2 + c^2 + d^2 = \frac{m}{n}。$$

求 $m + n$。

8. An ant makes a sequence of moves on a cube, where a move consists of walking from one vertex to an adjacent vertex along an edge of the cube. Initially the ant is at a vertex of the bottom face of the cube and chooses one of the three adjacent vertices to move to as its first move. For all moves after the first move, the ant does not return to its previous vertex, but chooses to move to one of the other two adjacent vertices. All choices are selected at random so that each of the possible moves is equally likely. The probability that after exactly 8 moves the ant is at a vertex of the top face of the cube is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

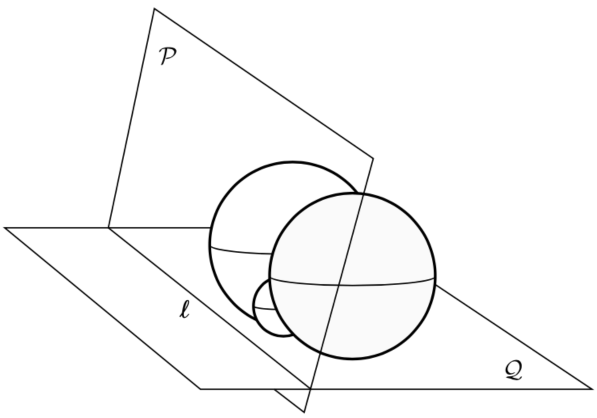
蚂蚁在立方体上进行一系列移动，其中每次移动是指从一个顶点沿着立方体的边走到相邻的顶点。起初，蚂蚁在立方体底面的某个顶点，并选择三个相邻的顶点之一做首次移动。对于首次移动以后的每次移动，蚂蚁不会返回它之前刚经过的顶点，而是选择移动到其他两个相邻顶点之一。所有的选择都是随机的，每次移动的各种可能出现的机会都相同。蚂蚁经过 8 次移动后，到达立方体顶面的顶点的概率为 $\frac{m}{n}$，其中 m 和 n 是互质的正整数。求 $m+n$。

9. Find the number of ordered pairs (m, n) such that m and n are positive integers in the set $\{1, 2, \cdots, 30\}$ and the greatest common divisor of 2^m+1 and 2^n-1 is not 1.

考虑有序数对 (m, n)，使得 m 和 n 是集合 $\{1, 2, \cdots, 30\}$ 中的正整数，并且 2^m+1 和 2^n-1 的最大公约数不是 1。求所有这样的有序数对的个数。

10. Two spheres with radii 36 and one sphere with radius 13 are each externally tangent to the other two spheres and to two different planes $\mathcal{P}$ and $\mathcal{Q}$. The intersection of planes $\mathcal{P}$ and $\mathcal{Q}$ is the line ℓ. The distance from line ℓ to the point where the sphere with radius 13 is tangent to plane $\mathcal{P}$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

两个半径为 36 的球和一个半径为 13 的球两两外切，并且与两个不同的平面 $\mathcal{P}$ 和 $\mathcal{Q}$ 相切。平面 $\mathcal{P}$ 和 $\mathcal{Q}$ 相交于直线 ℓ。从直线 ℓ 到半径为 13 的球与平面 $\mathcal{P}$ 的切点的距离为 $\frac{m}{n}$，其中 m 和 n 是互质的正整数。求 $m+n$。



11. A teacher was leading a class of four perfectly logical students. The teacher chose a set S of four integers and gave a different number in S to each student. Then the teacher announced to the class that the numbers in S were four consecutive two digit positive integers, that some number in S was divisible by 6, and a different number in S was divisible by 7. The teacher then asked if any of the students could deduce what S is, but, in unison, all of the students replied no.

    However, upon hearing that all four students replied no, each student was able to determine the elements of S. Find the sum of all possible values of the greatest element of S.

    课堂里有一位老师和四位能进行完备逻辑推理的学生。老师选择了一组由四个整数组成的集合 S，并给了每位学生一个 S 中的数，互不相同。然后老师向大家宣布 S 中的数是四个连续的两位正整数，S 中的某数能被 6 整除，而 S 中的另一个不同的数能被 7 整除。然后，老师问是否有任何学生可以推断出集合 S 的构成，但所有学生一致回答“否”。

    但是，在听到所有四位学生都回答“否”后，每位学生都能够确定出 S 中的元素。求 S 中最大元素的所有可能值的总和。

12. A convex quadrilateral has area 30 and side lengths $5, 6, 9$, and 7, in that order. Denote by θ the measure of the acute angle formed by the diagonals of the quadrilateral. Then $\tan\theta$ can be written in the form $\dfrac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

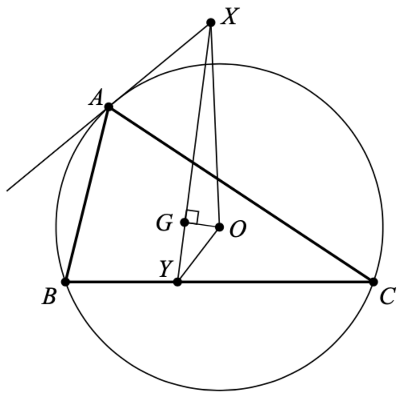
    某凸四边形的面积为 30，边长依次为 5，6，9，7。用 θ 表示由四边形的对角线所形成的锐角。$\tan\theta$ 可以写成 $\dfrac{m}{n}$ 的形式，其中 m 和 n 是互质的正整数。求 $m+n$。

13. Find the least positive integer n for which 2^n+5^n-n is a multiple of 1000 .

    求最小的正整数 n，使得 2^n+5^n-n 是 1000 的倍数。

14. Let $\triangle ABC$ be an acute triangle with circumcenter O and centroid G. Let X be the intersection of the line tangent to the circumcircle of $\triangle ABC$ at A and the line perpendicular to $\overline{GO}$ at G. Let Y be the intersection of lines XG and BC. Given that the measures of $\angle ABC, \angle BCA$, and $\angle XOY$ are in the ratio $13:2:17$, the degree measure of $\angle BAC$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

令 $\triangle ABC$ 是外心为 O 和重心为 G 的锐角三角形。设 X 为经过 A 的 $\triangle ABC$ 的外接圆的切线与经过 G 的垂直于 $\overline{GO}$ 的垂线的交点。设 Y 是 XG 和 BC 的交点。已知 $\angle ABC$，$\angle BCA$，$\angle XOY$ 的度数之比为 $13:2:17$，$\angle BAC$ 的度数可以写为 $\frac{m}{n}$，其中 m 和 n 是互质的正整数。求 $m+n$。



15. Let $f(n)$ and $g(n)$ be functions satisfying

$$f(n) = \begin{cases} \sqrt{n} & \text{if } \sqrt{n} \text{ is an integer} \\ 1 + f(n+1) & \text{otherwise} \end{cases}$$

and

$$g(n) = \begin{cases} \sqrt{n} & \text{if } \sqrt{n} \text{ is an integer} \\ 2 + g(n+2) & \text{otherwise} \end{cases}$$

for positive integers n. Find the least positive integer n such that $\frac{f(n)}{g(n)} = \frac{4}{7}$.

对于正整数 n，函数 $f(n)$ 和 $g(n)$ 满足：

$$f(n) = \begin{cases} \sqrt{n} & \text{当}\sqrt{n}\text{是整数时} \\ 1 + f(n+1) & \text{其他情形} \end{cases}$$

以及

$$g(n) = \begin{cases} \sqrt{n} & \text{当}\sqrt{n}\text{是整数时} \\ 2 + g(n+2) & \text{其他情形} \end{cases}$$

求最小的正整数 n 使得 $\frac{f(n)}{g(n)} = \frac{4}{7}$。

2021 AIME Ⅱ Answer Key

1.550	6.454	11.258
2.336	7.145	12.047
3.080	8.049	13.797
4.330	9.295	14.592
5.736	10.335	15.258