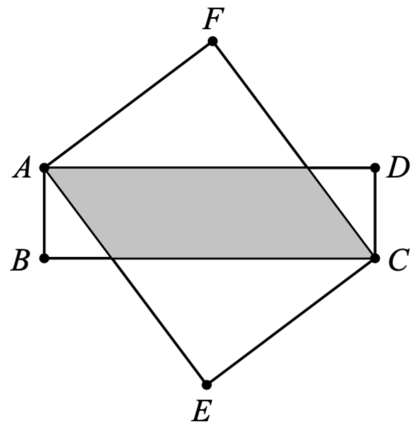
2021 AIME I Problems

1. Zou and Chou are practicing their 100-meter sprints by running 6 races against each other. Zou wins the first race, and after that, the probability that one of them wins a race is $\frac{2}{3}$ if they won the previous race but only $\frac{1}{3}$ if they lost the previous race. The probability that Zou will win exactly 5 of the 6 races is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

   小邹和小周两人进行了 6 场 100 米短跑比赛。小邹在第一场比赛中获胜，此后的每场比赛，在前一场比赛中获胜的人获胜的概率为 $\frac{2}{3}$，但在前一场比赛中失利的人获胜概率只有 $\frac{1}{3}$。小邹在 6 场比赛中恰好赢了 5 场的概率是 $\frac{m}{n}$，其中 m 和 n 是互质的正整数。求 $m+n$。

2. In the diagram below, $ABCD$ is a rectangle with side lengths $AB=3$ and $BC=11$, and $AECF$ is a rectangle with side lengths $AF=7$ and $FC=9$. The area of the shaded region common to the interiors of both rectangles is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

   在下图中，$ABCD$ 是边长为 $AB=3$ 和 $BC=11$ 的矩形，$AECF$ 是边长为 $AF=7$ 和 $FC=9$ 的矩形。两个矩形内部的公共区域如阴影部分所示，面积为 $\frac{m}{n}$，其中 m 和 n 为互质的正整数。求 $m+n$。



3. Find the number of positive integers less than 1000 that can be expressed as the difference of two integral powers of 2.

   求可以表示成为两个 2 的整数方幂之差的小于 1000 的正整数的个数。

4. Find the number of ways 66 identical coins can be separated into three nonempty piles so that there are fewer coins in the first pile than in the second pile and fewer coins in the second pile than in the third pile.

   将 66 枚相同的硬币分成三堆，每堆都有硬币，并且第一堆的硬币比第二堆的少，第二堆的硬币比第三堆的少。问这样的分法共有多少种？

5. Call a three-term strictly increasing arithmetic sequence of integers special if the sum of the squares of the three terms equals the product of the middle term and the square of the common difference. Find the sum of the third terms of all special sequences.

   对于一个由三个整数组成的严格递增的等差数列，如果三项的平方和等于中间项与公差平方的乘积，那么它称为特殊的。求所有特殊数列的第三项之和。

6. Segments $\overline{AB}, \overline{AC}$, and $\overline{AD}$ are edges of a cube and segment $\overline{AG}$ is a diagonal through the center of the cube. Point P satisfies $BP = 60\sqrt{10}$, $CP = 60\sqrt{5}$, $DP = 120\sqrt{2}$, and $GP = 36\sqrt{7}$. Find AP.

   线段 $\overline{AB}$，$\overline{AC}$ 和 $\overline{AD}$ 是一个立方体的边，而线段 $\overline{AG}$ 是通过立方体中心的对角线。点 P 满足 $BP = 60\sqrt{10}$，$CP = 60\sqrt{5}$，$DP = 120\sqrt{2}$，并且 $GP = 36\sqrt{7}$ 。求 AP 的长度。

7. Find the number of pairs (m, n) of positive integers with $1 \le m < n \le 30$ such that there exists a real number x satisfying

$$\sin(mx) + \sin(nx) = 2.$$

   求具有如下性质的正整数对 (m, n) 的数目：$1 \le m < n \le 30$，并且存在实数 x 满足

$$\sin(mx) + \sin(nx) = 2。$$

8. Find the number of integers c such that the equation

$$\left||20|x| - x^2| - c\right| = 21$$

   has 12 distinct real solutions.

   求使得方程

$$\left||20|x| - x^2| - c\right| = 21$$

   有 12 个不同的实数解的整数 c 的个数。

9. Let $ABCD$ be an isosceles trapezoid with $AD = BC$ and $AB < CD$. Suppose that the distances from A to the lines BC, CD, and BD are $15, 18$, and 10, respectively. Let K be the area of $ABCD$. Find $\sqrt{2} \cdot K$.

   设 $ABCD$ 为等腰梯形，$AD = BC$，并且 $AB < CD$。假设从 A 到直线 BC，CD，和 BD 的距离分别为 15，18 和 10。设 K 为 $ABCD$ 的面积。求 $\sqrt{2} \cdot K$ 的值。

10. Consider the sequence $(a_k)_{k\geq 1}$ of positive rational numbers defined by $a_1 = \dfrac{2020}{2021}$ and for $k \geq 1$, if $a_k = \dfrac{m}{n}$ for relatively prime positive integers m and n, then

$$a_{k+1} = \frac{m+18}{n+19}.$$

Determine the sum of all positive integers j such that the rational number a_j can be written in the form $\dfrac{t}{t+1}$ for some positive integer t.

由有理数组成的数列 $(a_k)_{k\geq 1}$ 按如下方式定义：$a_1 = \dfrac{2020}{2021}$，并且对于 $k \geq 1$，如果 $a_k = \dfrac{m}{n}$ 对于互质的正整数 m 和 n 成立，那么

$$a_{k+1} = \frac{m+18}{n+19}。$$

求能够找到正整数 t，使得有理数 a_j 可以写成 $\dfrac{t}{t+1}$ 形式的所有正整数 j 的总和。

11. Let $ABCD$ be a cyclic quadrilateral with $AB = 4, BC = 5, CD = 6$, and $DA = 7$. Let A_1 and C_1 be the feet of the perpendiculars from A and C, respectively, to line BD, and let B_1 and D_1 be the feet of the perpendiculars from B and D, respectively, to line AC. The perimeter of $A_1B_1C_1D_1$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

设 $ABCD$ 为圆内接四边形，$AB = 4$，$BC = 5$，$CD = 6$ 并且 $DA = 7$。设 A_1 和 C_1 分别是 A 和 C 到直线 BD 的垂线的垂足，设 B_1 和 D_1 分别是 B 和 D 到直线 AC 的垂线的垂足。$A_1B_1C_1D_1$ 周长为 $\dfrac{m}{n}$，其中 m 和 n 为互质的正整数。求 $m+n$。

12. Let $A_1A_2A_3 \ldots A_{12}$ be a dodecagon (12-gon). Three frogs initially sit at A_4, A_8, and A_{12}. At the end of each minute, simultaneously each of the three frogs jumps to one of the two vertices adjacent to its current position, chosen randomly and independently with both choices being equally likely. All three frogs stop jumping as soon as two frogs arrive at the same vertex at the same time. The expected number of minutes until the frogs stop jumping is $\dfrac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

设 $A_1A_2A_3 ... A_{12}$ 是一个十二边形。三只青蛙分别坐在 A_4、A_8 和 A_{12} 处。每分钟结束时，三只青蛙同时分别跳到与其当前位置相邻的两个顶点中的一个，这两个顶点随机独立的选择，可能性相同。只要有两只青蛙同时到达同一个顶点，那么所有三只青蛙就会停止跳跃。从开始到青蛙停止跳跃所需的分钟数的期望为 $\dfrac{m}{n}$，其中 m 和 n 为互质的正整数。求 $m+n$。

13. Circles ω_1 and ω_2 with radii 961 and 625, respectively, intersect at distinct points A and B. A third circle ω is externally tangent to both ω_1 and ω_2. Suppose line AB intersects ω at two points P and Q such that the measure of minor arc $\widehat{PQ}$ is 120°. Find the distance between the centers of ω_1 and ω_2.

圆 ω_1 和 ω_2 的半径分别为 961 和 625，两圆交于不同的点 A 和 B。第三个圆 ω 与 ω_1 和 ω_2 相外切。假设直线 AB 与 ω 相交于两点 P 和 Q，使得劣弧 $\widehat{PQ}$ 是 120°。求 ω_1 和 ω_2 的圆心之间的距离。

14. For any positive integer $a, \sigma(a)$ denotes the sum of the positive integer divisors of a. Let n be the least positive integer such that $\sigma(a^n) - 1$ is divisible by 2021 for all positive integers a. Find the sum of the prime factors in the prime factorization of n.

对于任何正整数 a，用 $\sigma(a)$ 表示 a 的正整数因数之和。令 n 是最小的正整数，使得对于所有的正整数 a，$\sigma(a^n) - 1$ 可以被 2021 整除。求 n 的质因数分解中所有质因数的和。

15. Let S be the set of positive integers k such that the two parabolas

$$y = x^2 - k \quad \text{and} \quad x = 2(y-20)^2 - k$$

intersect in four distinct points, and these four points lie on a circle with radius at most 21. Find the sum of the least element of S and the greatest element of S.

使得两条抛物线

$$y = x^2 - k \quad \text{和} \quad x = 2(y-20)^2 - k$$

相交于四个不同点，并且这四个点在某个半径不超过 21 的圆上的正整数 k 组成的集合记为 S。求 S 中最小元素与 S 中最大元素的和。

2021 AIME I Answer Key

1.097	6.192	11.301
2.109	7.063	12.019
3.050	8.057	13.672
4.331	9.567	14.125
5.031	10.059	15.285