



**MAA AMC**  
American Mathematics Competitions

**MAA American Mathematics Competitions**  
**42nd Annual**

**AIME II**

**American Invitational Mathematics Examination II**

**Wednesday, February 7, 2024**



## INSTRUCTIONS

1. DO NOT TURN THE PAGE UNTIL YOUR COMPETITION MANAGER TELLS YOU TO BEGIN.
2. This is a 15-question competition. All answers are integers ranging from 000 to 999, inclusive.
3. Mark your answer to each problem on the answer sheet with a #2 pencil. Check blackened answers for accuracy and erase errors completely. Only answers that are properly marked on the answer sheet will be scored.
4. SCORING: You will receive 1 point for each correct answer, 0 points for each problem left unanswered, and 0 points for each incorrect answer.
5. Only blank scratch paper, rulers, compasses, and erasers are allowed as aids. Prohibited materials include calculators, smartwatches, phones, computing devices, protractors, and graph paper.
6. Figures are not necessarily drawn to scale.
7. You will have 3 hours to complete the competition once your competition manager tells you to begin.

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The problems and solutions for this AIME were prepared by the  
MAA AIME Editorial Board under the direction of:  
Jonathan Kane and Sergey Levin, co-Editors-in-Chief

The MAA AMC Office reserves the right to disqualify scores from a school if it determines that the rules or the required security procedures were not followed.

The publication, reproduction, or communication of the problems or solutions of this competition during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via phone, email, or digital media of any type during this period is a violation of the competition rules.

*A combination of your AIME score and your AMC 10/12 score is used to determine eligibility for participation in the USA (Junior) Mathematical Olympiad.*

**Problem 1:**

Among the 900 residents of Aimeville, there are 195 who own a diamond ring, 367 who own a set of golf clubs, and 562 who own a garden spade. In addition, each of the 900 residents owns a bag of candy hearts. There are 437 residents who own exactly two of these things, and 234 residents who own exactly three of these things. Find the number of residents of Aimeville who own all four of these things.

**Problem 2:**

A list of positive integers has the following properties:

- The sum of the items in the list is 30.
- The unique mode of the list is 9.
- The median of the list is a positive integer that does not appear in the list itself.

Find the sum of the squares of all the items in the list.

**Problem 3:**

Find the number of ways to place a digit in each cell of a  $2 \times 3$  grid so that the sum of the two numbers formed by reading left to right is 999, and the sum of the three numbers formed by reading top to bottom is 99. The grid below is an example of such an arrangement because  $8 + 991 = 999$  and  $9 + 9 + 81 = 99$ .

0	0	8
9	9	1

**Problem 4:**

Let  $x$ ,  $y$ , and  $z$  be positive real numbers that satisfy the following system of equations:

$$\begin{aligned}\log_2 \left( \frac{x}{yz} \right) &= \frac{1}{2} \\ \log_2 \left( \frac{y}{xz} \right) &= \frac{1}{3} \\ \log_2 \left( \frac{z}{xy} \right) &= \frac{1}{4}.\end{aligned}$$

Then the value of  $\left| \log_2 (x^4 y^3 z^2) \right|$  is  $\frac{m}{n}$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .

**Problem 5:**

Let  $ABCDEF$  be a convex equilateral hexagon in which all pairs of opposite sides are parallel. The triangle whose sides are extensions of segments  $\overline{AB}$ ,  $\overline{CD}$ , and  $\overline{EF}$  has side lengths 200, 240, and 300. Find the side length of the hexagon.

**Problem 6:**

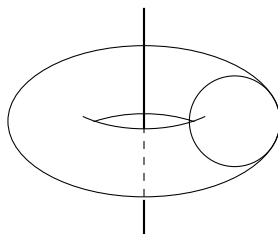
Alice chooses a set  $A$  of positive integers. Then Bob lists all finite nonempty sets  $B$  of positive integers with the property that the maximum element of  $B$  belongs to  $A$ . Bob's list has 2024 sets. Find the sum of the elements of  $A$ .

**Problem 7:**

Let  $N$  be the greatest four-digit positive integer with the property that whenever one of its digits is changed to 1, the resulting number is divisible by 7. Let  $Q$  and  $R$  be the quotient and the remainder, respectively, when  $N$  is divided by 1000. Find  $Q + R$ .

**Problem 8:**

Torus  $\mathcal{T}$  is the surface produced by revolving a circle with radius 3 around an axis in the plane of the circle that is a distance 6 from the center of the circle.



Let  $\mathcal{S}$  be a sphere with radius 11. When  $\mathcal{T}$  rests on the inside of  $\mathcal{S}$ , it is internally tangent to  $\mathcal{S}$  along a circle with radius  $r_i$ , and when  $\mathcal{T}$  rests on the outside of  $\mathcal{S}$ , it is externally tangent to  $\mathcal{S}$  along a circle with radius  $r_o$ . The difference  $r_i - r_o$  can be written as  $\frac{m}{n}$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .

**Problem 9:**

There is a collection of 25 indistinguishable black chips and 25 indistinguishable white chips. Find the number of ways to place some of these chips in 25 unit cells of a  $5 \times 5$  grid so that

- each cell contains at most one chip,
- all chips in the same row and all chips in the same column have the same color, and
- any additional chip placed on the grid would violate one or more of the previous two conditions.

**Problem 10:**

Let  $\triangle ABC$  have incenter  $I$ , circumcenter  $O$ , inradius 6, and circumradius 13. Suppose that  $\overline{IA} \perp \overline{OI}$ . Find  $AB \cdot AC$ .

**Problem 11:**

Find the number of triples of nonnegative integers  $(a, b, c)$  satisfying  $a + b + c = 300$  and

$$a^2b + a^2c + b^2a + b^2c + c^2a + c^2b = 6,000,000.$$

**Problem 12:**

Let  $O(0, 0)$ ,  $A(\frac{1}{2}, 0)$ , and  $B(0, \frac{\sqrt{3}}{2})$  be points in the coordinate plane. Let  $\mathcal{F}$  be the family of segments  $\overline{PQ}$  of unit length lying in the first quadrant with  $P$  on the  $x$ -axis and  $Q$  on the  $y$ -axis. There is a unique point  $C$  on  $\overline{AB}$ , distinct from  $A$  and  $B$ , that does not belong to any segment from  $\mathcal{F}$  other than  $\overline{AB}$ . Then  $OC^2 = \frac{p}{q}$ , where  $p$  and  $q$  are relatively prime positive integers. Find  $p + q$ .

**Problem 13:**

Let  $\omega \neq 1$  be a 13th root of unity. Find the remainder when

$$\prod_{k=0}^{12} (2 - 2\omega^k + \omega^{2k})$$

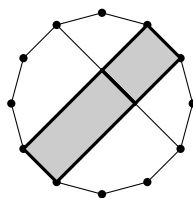
is divided by 1000.

**Problem 14:**

Let  $b \geq 2$  be an integer. Call a positive integer  $n$  *b-eautiful* if it has exactly two digits when expressed in base  $b$ , and these two digits sum to  $\sqrt{n}$ . For example, 81 is 13-eautiful because  $81 = \underline{6}\underline{3}_{13}$  and  $6 + 3 = \sqrt{81}$ . Find the least integer  $b \geq 2$  for which there are more than ten *b-eautiful* integers.

**Problem 15:**

Find the number of rectangles inside a fixed regular dodecagon (12-gon) where each side of the rectangle lies on a side or on a diagonal of the dodecagon. The diagram below shows three of those rectangles.



1. 073
2. 236
3. 045
4. 033
5. 080
6. 055
7. 699
8. 127
9. 902
10. 468
11. 601
12. 023
13. 321
14. 211
15. 315