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INDEPENDENT SETS IN THE NEIGHBORHOOD SYSTEMS OF BALANCED BICLIQUES: OPTIMIZATION AND POLYNOMIAL REPRESENTATIONS

Rosalio G. Artes Jr.[1], Regimar A. Rasid[1], Sherna A. Rasid[2], Bayah J. Amiruddin-Rajik[1] and Al-Jayson U. Abubakar[1]

[1]Mindanao State University
Tawi-Tawi College of Technology and Oceanography
Sanga-Sanga, 7500 Bongao, Tawi-Tawi
Philippines
e-mail: rosalioartes@msutawi-tawi.edu.ph
regimarrasid@msutawi-tawi.edu.ph
bayahamiruddin@msutawi-tawi.edu.ph
al-jaysonabubakar@msutawi-tawi.edu.ph

[2]Tawi-Tawi School of Arts and Trade
Schools Division of Tawi-Tawi
Ministry of Basic, Higher, and Technical Education
Tawi-Tawi, Philippines
e-mail: shernrasid@gmail.com

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Abstract

In this paper, we determine the cardinality of an optimal independent set in the neighborhood system of a balanced biclique in a graph. We introduce a bivariate polynomial which represents the number of balanced bicliques corresponding to the cardinalities of maximum independent sets. Finally, we establish the explicit forms of the balanced biclique independent neighborhood polynomials of some special graphs.

1. Introduction

Graph polynomials surfaced recently in the field of discrete and applied mathematics because of its applications in Chemistry, Biology, and Physics [13]. Artes and Rasid [9] introduced the concept of balanced biclique polynomials of graphs and obtained some properties. Such a polynomial counts the number of balanced bicliques in a graph. The corresponding polynomials for some special graphs are investigated. Moreover, results on this polynomial for the join and the corona of graphs are obtained in [10]. All of these graph polynomials are univariate.

The idea of neighborhood polynomial was pioneered by Artes et al. [15]. Several two-variable polynomials had been investigated since then which involved counting the number of substructures with corresponding neighborhood system cardinality considering optimality conditions. Such polynomials include the connected dominating independent neighborhood polynomials [1], convex independent common neighborhood polynomials [2], connected common neighborhood polynomials of cliques in some special graphs [3], in the join of graphs [4], in the corona of graphs [5], biclique neighborhood polynomials [6], connected total dominating neighborhood polynomials [7], and geodetic subdominating neighborhood polynomials [8]. Balanced biclique common neighborhood polynomials of some special graphs are obtained in [11]. Also, results on the clique neighborhood polynomial of a graph are obtained in [12].

In this paper, we introduce a new bivariate polynomial considering the balanced biclique as our baseline substructure and the maximal independent sets of its neighborhood systems.

For a connected graph G, a subset of $V(G)$ is a biclique if it induces a complete bipartite subgraph of G. A biclique is said to be *balanced biclique* if the partitions are equivalent. For $q \geq 3$, we define a complete *q-partite graph* as a graph with q independent vertex partitions such that every vertex in a partition is connected by an edge to all other vertices in other partitions.

The *balanced biclique independent neighborhood polynomial* of G is given by

$$\Gamma_{bin}(G;\ x,\ y) = \sum_{j=0}^{n-2i} \sum_{i=1}^{\frac{\beta(G)}{2}} b_{ij}(G)x^{2i}y^{j},$$

where $b_{ij}(G)$ is the number of balanced bicliques of G of order $2i$ with maximum independent neighborhood systems having cardinality equal to j and $\beta(G)$ is the cardinality of a maximum balanced biclique of G. By a *maximum independent neighborhood* of $S \subseteq V(G)$, we mean a maximum independent subset of $N_G(S)$. That is, an independent subset of $N_G(S)$ of maximum cardinality.

The balanced biclique independent neighborhood polynomials of cycles and complete graphs are found in the following section. We present some characterizations from [9] that will guide us in developing our results.

2. Results on Cycles and Complete Graphs

The result below characterizes a balanced biclique in a cycle.

Lemma 2.1 [9]**.** *For a natural number* $n \geq 5$, *a balanced biclique is only that which induces a* $K_{1,1}$.

The polynomial for C_n is established in the following theorem.

Theorem 2.2. *Let* $n \geq 5$. *Then* $\Gamma_{bin}(C_n;\ x,\ y) = nx^2y^2$.

Proof. Let S be a balanced biclique of C_n. Then $\langle S \rangle = K_{1,1}$. Simple counting techniques considering the adjacency property of C_n lead to the desired result. □

For complete graphs, the adjacency property leads us to the conclusion that balanced bicliques are exactly those that induce a $K_{1,1}$. Moreover, the maximum independent sets of the neighborhood system follow from the completeness (adjacency property) of a complete graph. This gives $j = 1$. Hence, we have the following result.

Theorem 2.3. *Let* n *be a natural number greater than one. Then*, $\Gamma_{bin}(K_n;\ x,\ y) = \binom{n}{2}x^2y$.

Proof. It is clear that the balanced bicliques in K_n are exactly $K_{1,1}$'s. By combinations, there are $\binom{n}{2}$ of them. By the adjacency property of K_n, the maximum independent neighborhood system for every biclique is 1. The polynomial follows immediately. □

The next section establishes the balanced biclique independent neighborhood polynomial of the complete bipartite graph $K_{m,n}$ and the complete q-partite graph $K_{r_1, r_2, \ldots, r_q}$.

3. Results on Bipartite and q-partite Graphs

This section presents the results for bipartite and complete bipartite graphs. We used the results from [9] as presented in the following characterization.

Lemma 3.1 [9]. *A subset S of* $V(K_{m,n})$ *induces a balanced biclique in* $K_{m,n}$ *if and only if* $S = S_1 \cup S_2$, *where* $S_1 \subseteq V(\overline{K_m})$ *and* $S_2 \subseteq V(\overline{K_n})$ *with* $|S_1| = |S_2|$.

With the above characterization, the following is established.

Theorem 3.2 [9]. *For* $m \leq n$, $b(K_{m,n}, x) = \sum_{i=1}^{m} \binom{m}{i}\binom{n}{i} x^{2i}$.

The polynomial for complete bipartite graph is established in the following theorem.

Theorem 3.3. *Let m and n be natural numbers satisfying* $m \leq n$. *Then*

$$\Gamma_{bin}(K_{m,n};\, x,\, y) = \sum_{i=1}^{m} \binom{m}{i}\binom{n}{i} x^{2i} y^{n-i}.$$

Proof. Let S be a balanced biclique of $K_{m,n}$. Then $S = S_1 \cup S_2$, where $|S_1| = S_2$ such that $S_1 \subseteq V_1$ and $S_2 \subseteq V_2$. Note that for each i, there are exactly $\binom{m}{i}\binom{n}{i}$ bicliques $K_{m,n}$ of order $2i$. The remaining vertices correspond to the independent neighborhood system, which is equal to $n - 2i$. The result follows. □

The balanced bicliques in the complete q-partite graph are characterized in the following result.

Lemma 3.4 [9]. *A subset S of* $V(K_{r_1, r_2, \ldots, r_q})$ *with independent vertex partition* $V_{r_1}, V_{r_2}, \ldots, V_{r_q}$ *induces a balanced biclique in* $K_{r_1, r_2, \ldots, r_q}$ *if and only if* $S = S_i \cup S_j$, *where* $S_i \subseteq V_{r_i}$ *and* $S_j \subseteq V_{r_j}$ *satisfying* $|S_i| = |S_j|$.

Finally, we establish the result for a q-partite graph in the following theorem.

Theorem 3.5. *Let* $\langle r_1, r_2, \ldots, r_q \rangle \subseteq \mathbb{Z}^+$ *such that* $r_1 \leq r_2 \leq \cdots \leq r_q$. *Then*

$$\Gamma_{bin}(K_{r_1, r_2, \ldots, r_q}; x, y) = \sum_{k=1}^{r_i} \sum_{i<j} \sum_{i=1}^{q-1} \binom{r_i}{k} \binom{r_j}{k} x^{2k} y^{\max\{r_j - k, r_s : s \neq i, j\}}.$$

Proof. Let S be a balanced biclique of $K_{r_1, r_2, \ldots, r_q}$. Then $S = S_1 \cup S_2$ with $|S_1| = S_2$ and $S_1 \subseteq V_s$, $S_2 \subseteq V_t$ for some $s, t \in \{1, 2, \ldots, q\}$. Note that there are $\sum_{i<j} \sum_{i=1}^{q-1} \binom{r_i}{k} \binom{r_j}{k}$ of these balanced bicliques of order $2k$ in $K_{r_1, r_2, \ldots, r_q}$ for each $k \in \{1, 2, \ldots, r_i\}$. Now, the maximum independent subsets must be contained entirely in a partite set by the adjacency property of the complete q-partite graph. This is exactly equal to the maximum of $r_j - k$ and $\max\{r_s : s \neq i, j\}$ by the monotonicity property of the sequence $\langle r_i \rangle_{i=1}^{q}$. Hence,

$$\Gamma_{bin}(K_{r_1, r_2, \ldots, r_q}; x, y) = \sum_{k=1}^{r_{q-1}} \sum_{i<j} \sum_{i=1}^{q-1} \binom{r_i}{k} \binom{r_j}{k} x^{2k} y^{\max\{r_j - k, r_s : s \neq i, j\}}.$$

This completes the proof. □

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