Expected Value Optimization, Client-Centric Planning, & Growth Optimal Selection: A Centuries-long, Three-Braided Conceptual Chronology

François Gadenne

October 6, 2020 (Updated Oct. 13; Nov. 13; Nov. 25, 2020)

Abstract

The roots of (i) expected value optimization, (ii) client-centric planning, and (iii) growth optimal selection twist and turn around one another over a centuries-long research history. This paper traces each strand chronologically. It also highlights their interconnections. Its goal is to connect what practitioners already know with ergodicity economics, because it is the next wave of product development and client advice.

We cannot individually experience ensemble averages, thus using expected value as the default investment decision criterion has created a large catalog of empirical puzzles, paradoxes, and anomalies. Financial economics uses mathematics as a language to conjure the structure of investment problems. Ergodicity economics (EE) restores mathematics as a method of skepticism to question the structure of economic and investment problems. This change has created a growing catalog of solutions to the traditional list of empirical puzzles, paradoxes, and anomalies.

The formalization of embedding randomness in time by Ole Peters in 2011, as an alternative to embedding randomness in the ensemble, is a critical development for the financial industry. At a personal level, it clarified the conceptual meaning of a path through three start-ups over three decades. This path went from expected value portfolio optimization, to client-centric retirement planning, and now growth-optimal product selection.

Following EE's example, the rapidly growing number of research papers becomes more manageable when one starts with the foundational papers. Thus, this narrated conceptual chronology starts each conceptual entry with the earliest research papers available and lists the matching entries in ascending chronological order.

This narrated conceptual chronology proved to be a helpful exercise to connect the dots between financial economics and ergodicity economics. It also proved to be a productive platform for the development of additional content: Identifying and rank-ordering the empirical puzzles, paradoxes, and anomalies of financial economics that matter the most for practitioners and that could benefit from solutions based on ergodicity economics.

Key Words: Financial economics, ergodicity economics, expected value, retirement planning, time average, portfolio optimization, client-centric, product selection

Introduction

The roots of (i) expected value optimization, (ii) client-centric planning, and (iii) growth optimal selection twist and turn around one another over a centuries-long research history. This paper traces each strand chronologically. It also highlights their interconnections. Its goal is to connect what practitioners already know with ergodicity economics because it is the next wave of product development and client advice.

We cannot individually experience ensemble averages, thus using expected value as the default investment decision criterion has created a large catalog of empirical puzzles, paradoxes, and anomalies. Financial economics uses mathematics as a language to conjure the structure of investment problems. Ergodicity economics restores mathematics as a method of skepticism to question the structure of economic and investment problems. This change has created a growing catalog of solutions to the traditional list of empirical puzzles, paradoxes, and anomalies.

The inspiration for this narrated conceptual chronology came from reading:

- Chapter 5 of Emanuel Derman's *Models.Behaving.Badly*. (2011)
- Ole Peters' and Alexander Adamou's course notes "Ergodicity Economics" (2018/2019), and matching research papers, as well as,
- Mark Kirstein's PhD thesis "The Ergodicity Problem in Economics" (2019/2020).

This narrated conceptual chronology proved to be a helpful exercise to connect the dots between financial economics and ergodicity economics. It also proved to be a productive platform for the development of additional content: Identifying and rank-ordering the empirical puzzles, paradoxes, and anomalies of financial economics that matter the most for practitioners, and that could benefit from solutions based on ergodicity economics.

Answers to Questions from Readers and Reviewers

The formalization of embedding randomness in time by Ole Peters in 2011, as an alternative to embedding randomness in the ensemble, is a critical development for the financial industry. At a personal level, it clarified the conceptual meaning of a path through three start-ups over three decades. This path went from expected value portfolio optimization to client-centric retirement planning and now growth optimal product selection.

Sometimes, the best way to articulate a perspective-changing idea, and why it is important to share this personal experience with others, is to present the questions (and the answers) that it triggered with readers and reviewers over the course of its development.

Here is a selection of such questions and answers to help place this narrated conceptual chronology (NCC) in its goal-oriented context:

Who is the audience for the NCC?

Financial advisors as well as insurance and investment product developers seeking to connect the dots between what they already know and ergodicity economics.

What is the justification for the NCC's three-braided conceptualization?

Reading financial research over 30 years as the basis to launch several start-ups, including:

- Rational Investors, a dot-com robo advisor focused on expected value portfolio optimization and sold to S&P in 1999,
- The Retirement Income Industry Association (RIIA) focused on the development of a professional designation for retirement planning, and sold to the Investments & Wealth Institute (formerly IMCA) in 2017,
- The Curve, Triangle & Rectangle Institute℠ (CTRI℠), a membership-based R&D institute that is working on a variety of topics, including EE.

Why call this a narrated chronology?

The conceptual entries in the chronology tell a story augmented with timeline era commentaries, entry summaries, and connections between entries.

How does the NCC fit in with the geometric analogies of CTRI?

EE adds a fourth geometric figure as follows:

- The Curve represents the efficient frontier of expected value portfolio optimization,
- The Triangle illustrates the Maslowian hierarchy often used to illustrate recommendations based on behavioral finance,
- The Rectangle matches the household balance sheet used in client-centric retirement planning, and
- The Circle symbolizes the unbroken symmetry of ergodic observables.

Why is NCC a three-braided story instead of a four-part story that uses the geometric analogies?

While it would be tempting to keep CTRI's geometric analogies, the development of NCC showed that the thesis of expected value as a decision criterion created a client-centric anti-thesis. The Triangle (behavioral finance) and the Rectangle (asset/liability matching) are part of this client-centric anti-thesis, a multi-faceted reaction against the empirical puzzles, paradoxes, and anomalies of expected value. Ergodicity economics has the potential to become the synthesis in the dialectic of Financial Economics research.

A Narrated Conceptual Chronology (NCC)

Following EE's example, the rapidly growing number of research papers becomes more manageable when one starts with the foundational papers. Thus, this narrated conceptual chronology starts each conceptual entry with the earliest research papers available, and lists the matching entries in ascending chronological order.

The 59 conceptual entries in this chronology are structured into four timeline eras as follows:

- The 19th century, and earlier
  - 8 entries
- The 20th century, through World War II
  - 7 entries
- The 20th century, post-World War II
  - 30 entries
- The 21st century
  - 14 entries

The twists and turns in the chronology make it necessary to create visual differentiation as follows:

- Timeline era commentaries are shown in *italic font*,
- Expected value entries are in normal font,
- Client-centric planning entries are in **green bold font**, and
- Growth optimal entries are in **black bold font**.

Additionally:

- Books are shown in *italics* without quote marks,
- Papers are in "normal font" with quote marks,
- Definitions are in [smaller Times New Roman font] in square brackets.

The 20th Century and Earlier

Timeline Era Commentary

It is remarkable that the distinction between the arithmetic mean and the geometric mean starts with Euclid. This distinction is the oldest conceptual entry in the chronology, and it is also one of the key differences between financial economics, and ergodicity economics. Another critical feature appears early on. The importance of the logarithm function is discovered in the 18th century. Further, and based on an effort to simplify complicated calculations in statistical mechanics, the concept of ergodicity emerges in the 19th century. Finally, even though both mathematical means appeared at the beginning of this chronology we observe that expected value dominated the geometric mean early on, perhaps on account of its simpler mathematics. As the 20th century starts, the stage is set for continued work on understanding the meaning of the means based on the microstructure of the data. However, it will take another 99 years before the meaning of the means becomes clear enough to offer generalized solutions beyond the expected value.

The inequality of the arithmetic and geometric means, starting in the 3rd century BCE with Euclid in "Book V, Proposition 25": Ἐὰν τέσσαρα μεγέθη ἀνάλογον ᾖ, τὸ μέγιστον [αὐτῶν] καὶ τὸ ἐλάχιστον δύο τῶν λοιπῶν μείζονά ἐστιν. "If four magnitudes are proportional then the (sum of the) largest and the smallest [of them] is greater than the (sum of the) remaining two (magnitudes)." A special case occurs when the middle terms are the same, in which case this proposition gives us a corollary: The arithmetic mean of two magnitudes is greater than their geometric mean. This inequality of means has been known for a long time.

Expected wealth decision criterion, starting in 1657 with the publication of Christiaan Huygens's 20-page tract *"*Libellus De ratiociniis in ludo alea*"* ("The Value of all Changes in Games of Fortune"). This was privately developed earlier by Blaise Pascal & Pierre de Fermat in 1654 in response to a question posed by Antoine Gombaud, Chevalier to Méré, about the validity of an even-money betting rule for a gambling game based on throwing a pair of dice 24 times. It was explicitly defined by Pierre-Simon Laplace as "L'Esperance Mathematique" in his book *La Théorie analytique des probabilités* (*Analytical Theory of Probabilities*) started in 1795 and published in 1812, and formalized with the notation of "E[X]" by William Allen Whitworth in 1901 in the fifth edition of *Choice and Chance, with 1000 exercises*.

Expected utility of wealth decision criterion, starting in 1710 with the St. Petersburg paradox as articulated in letters between Nikolaus Bernoulli and Pierre Remond de Montmort. The letters were triggered by the publication of Montmort's book *Essay d'analyse sur les jeux de hazard* in 1708, and documented in the second edition of the book published in 1714. The solution was first presented in Daniel Bernoulli's book *Specimen Theoriae Novae de Mensura Sortis* (*Exposition of a New Theory on the*

Measurement of Risk) published in 1783. The solution used the expected utility of wealth decision criterion, also known as the "moral expectation," and the logarithm of wealth. This solution was improved by Laplace in his book published in 1812 where he corrected a mathematical error in Daniel Bernoulli's solution.

<u>Geometric mean decision criterion</u> as an alternative solution to the St. Petersburg paradox, starting with Leonhard Euler (likely written in 1730) published in *Opera posthuma [E811]* in 1862, and followed-up by William Allen Whitworth in 1870 with the second edition of his book *Choice and Chance*. Euler may not have published E811, based on the then consensus opinion that Bernoulli's expected utility solution was the better solution.

<u>Conditional probabilities</u>, starting with Thomas Bayes' "An Essay towards solving a Problem in the Doctrine of Chances" that was read to the Royal Society in 1763 after his death, showing that probabilistic boundaries could be placed on uncertain events, and formalized in 1814 with Laplace's book *Essai philosophique sur les probabilités* (*Philosophical Essay on Probabilities*) showing [Principle VI: Principle of Insufficient Reason] as a method of statistical inference to update the probability for a hypothesis as more information becomes available. This sets the stage for multifaceted interpretations of the nature of probabilities.

<u>The law of one price</u>, starting in 1838 with Antoine Augustin Cournot's book *Recherches sur les principes mathématiques de la théorie des richesses* (*Researches on the Mathematical Principles of the Theory of Wealth*), and followed up in 1919 by Alfred Marshall's book *Industry and Trade*. This is a valuation technique by analogy: To know the value of a security, use the known price of another, but similar security.

<u>Rational homo economicus</u>, starting in 1883 with Charles Stanton Devas' book *Groundwork of Economics* in a critique of John Stuart Mill's 1848 book *Principles of Political Economy*. This is the conceptualization of a (i) bias-free, (ii) rational, (iii) utility-maximizing decision-maker, with (iv) unlimited cognitive capacity, (v) perfect access to information, (vi) narrow self-interest, and (vii) preference consistency. These cognitive, psychological, and social simplifications further enable the mathematical treatment of the problem of optimal decision making under uncertainty.

<u>The statistical ensemble view and the ergodicity "Kunstgriff,"</u> starting in 1896 with Ludwig Eduard Boltzmann's two-volume book *Vorlesungen über Gastheorie* (*Lectures on Gas Theory*), and continued in 1902 with Josiah Willard Gibbs' book *Elementary Principles in Statistical Mechanics*. In 1912 Paul and Tatiana Ehrenfest's book *Begriffliche Grundlagen der Statistischen Auffassung in der Mechanik* (*The Conceptual Foundations of the Statistical Approach in Mechanics*), a reconstruction of Boltzmann's work, emphasized the fundamental role for the ergodic hypothesis: If true, then averaging over the long time would be identical to phase averaging with the

microcanonical distribution (the ensemble). At the same time they expressed doubts about its validity beyond a "kunstgriff," a mathematical trick.

The 20th Century through World War II

Timeline Era Commentary

The first half of the 20th century axiomatized the mathematics of expected value optimization, introducing simplifying modeling assumptions. This period also introduced the first client-centric entries in this chronology. The difference and tension between the theoretician observer and the empirical decision-maker remained unresolved without an appeal to unconstrained psychological reasons until the advent of ergodicity economics.

The random walk model and Brownian motion, starting in 1900 with Louis Bachelier's thesis on the statistical analysis of periodic returns with the view that market prices change randomly, thus justifying the use of the law of large numbers [the average from a large number of trials should approximate the expected value of the population and should become closer to with additional trials], and the central limit theorem [the sum of normalized, independent random variables approximates a normal distribution, even when they are not originally normally distributed]. This was followed in 1953 by Maurice Kendall's paper "The Analysis of Economic Time Series, Part 1: Prices," and in 1964 by Paul Cootner's book *The Random Character of Stock Market Prices*, as well as Eugene Fama's PhD thesis and its matching 1965 article "Random Walks in Stock Market Prices," and finally Burton Malkiel's 1973 book *A Random Walk Down Wall Street* – currently in its 12th edition. Combining the random walk model with Cournot's law of one price means that two securities with the same volatility should be expected to have the same expected return.

Axiomatization of subjective probabilities and subjective expected utility, starting with "Truth and Probability," a 36-page essay written in 1926 (published in 1931) by Frank Ramsey, and axiomatized by Leonard Jimmie Savage in 1954 with his book *The Foundations of Statistics*. Answering questions such as [Are probabilities objective characteristics of what is observed, or are they subjective judgments of the observer? Are they a measure of the observer's ignorance about the observed? How can we derive a consistent theory of choice from such subjective probabilities?], **this continues the development of fundamental questions about the meaning of probabilities and the difference between the theoretician observer and the empirical decision-maker.**

Statistical inferences for rational decision-making, starting in 1933 with Jerzy Neyman and E.S. Pearson's paper "On the Problem of the Most Efficient Tests of Statistical Hypotheses" [Hypothesis Testing with Type I/Type II errors], as contrasted with Sir Ronald Fisher's 1925 book *Statistical Methods for Research Workers* [Confidence Interval Testing with p-values]. The mechanical application of these two related, yet different, methods of induction has led to a systemic problem of reproducibility in fields with high clinical ambiguity such as the social sciences.

The axiomatization of objective probabilities and the interpretations of probabilities, starting in 1933 with Andrei Kolmogorov's book *Foundations of the Theory of Probability* and followed-up with formalized interpretations of the nature of probabilities that include: (i) the classical, as in games of chance with finite possible outcomes, (ii) the frequentist, as in relative frequency over time to infinity, (iii) the subjective (Bayesian), as in degrees of belief in games of credence, and (iv) the propensity, as in physical degrees of causal connection.

Decision theory, normative, descriptive, or optimal game-based decisions under constraints and uncertainty with the help of statistical methods, starting with Abraham Wald in 1939 with his paper "Contribution to the Theory of Statistical Estimation and Testing Hypotheses" pointing out that statistical hypothesis testing and parameter estimations are parts of a larger set of decisions, and in 1950 when E. L. Lehman first used the expression "decision theory" in his paper "Some Principles of the Theory of Testing Hypotheses."

Matching asset and liability durations with dedicated portfolio theory, starting in 1942 with Tjalling Charles Koopmans in his paper "The Risk of Interest Fluctuations in Life Insurance Companies." This developed into an institutional practice starting in the early 1970s. It became a retail practice in 2005 with Steve Huxley and Brent Burns' book "Asset Dedication." This marks an important transition from institutional asset management to household asset management.

Minimax/maximin decision rule, starting in 1945 with Abraham Wald's paper "Statistical decision functions which minimize the maximum risk." [Originally, it was a two-player, zero-sum game decision-rule to minimize a possible loss from the worst-case scenario (minimax), or to maximize the gain from the lowest gain scenario (maximin)**.] It delivers a resilient order of conclusions due to non-probabilistic scenario analysis based on outcomes that can be compared and ranked (ordinal) as opposed to probabilistic expected values and utilities that return a specific number (cardinal). This marks the beginning of game-theory – an attempt to arrive at optimal decision rules based on the rational behavior of participants, under all possible scenarios.**

The 20th Century, Post-World War II

Timeline Era Commentary

The second-half of the 20th century started with the formalization of mathematics as a language that defines the structure of economic problems. In particular, asserting the ergodicity of economic observables enshrined expected value as the decision criterion. This created a large catalog of empirical puzzles, paradoxes, and anomalies that drove many researchers to look for solutions with unconstrained psychology explanations, before and instead of time average growth rate optimization.

Expected utility theory (EUT) axiomatization, a descriptive and prescriptive, neo-classical economics theory of rational individual (homo economicus) choice under conditions of risk and uncertainty, starting in 1947 with Von Neumann and Morgenstern's book *Theory of Games and Economic Behavior*. This work formalizes four "necessary and sufficient" conditions for the theory to hold. These axioms of rational decision-making include: (i) completeness of preferences and choices, (ii) consistent (transitive) choices, (iii) independence of preferences mixed with irrelevant alternatives, and (iv) the continuity of combinations of choices. Empirical departures from these axioms of rational choice came to be viewed by the theoretician observers as biases and errors on the part of the empirical decision-makers.

The ergodic axiom in economics, starting in 1947 with Paul Samuelson's book *Foundations of Economic Analysis* and its front-page quote from J. Willard Gibbs: "Mathematics is a language." It is Samuelson that moved the ergodic hypothesis from physics to economics, to create tractable mathematics of equilibrium. This completed the lineage from Ludwig Eduard Blotzmann's H-Theorem in his 1872 paper "Weitere Studien über das Wärmegleichgewicht unter Gasmolekülen" ("Further Studies on the Thermal Equilibrium of Gas Molecules"), to J. Willard Gibbs's 1902 book *Elementary Principles in Statistical Mechanics, Developed with Especial Reference to the Rational Foundations of Thermodynamics*, and to Gibb's student and disciple as well as Samuelson's professor of mathematics at Harvard, Edwin B. Wilson. This ergodic hypothesis at the core of economics became largely forgotten and unexamined. This assumption of the ergodic axiom in open-systems leads to the ergodic fallacy, the false belief in economic and financial relationships that do not exist, or that change constantly. This is a root-cause for many puzzles, paradoxes, and anomalies in financial economics.

Growth optimal leverage, starting in 1948 with Claude Shannon's paper "A Mathematical Theory of Communication," and formalized in 1956 with John L. Kelly's paper "A New Interpretation of Information Rate," as a decision criterion for intertemporal portfolio choice. It focuses on optimal growth rates based on wealth allocation to the risky asset by maximizing the expected logarithm of wealth. This is a

special case, in the ergodicity economics framework, where the growth dynamic is multiplicative, and the transformation function is the logarithm.

Style investing, starting in 1949 with Benjamin Graham's book *The Intelligent Investor*, focusing on the buy-and-hold, value style, and differentiating "Mr. Market's" daily price from the investor's empirical analysis of the value of the stock to determine the presence of a "margin of safety." Style investing developed beyond value investing to include Morningstar's "style box," factor investing, alternative investments, and ESG/SRI/impact investing.

Modern Portfolio Theory (MPT), starting in 1952 with Harry Markowitz's papers "Portfolio Selection" and "The Utility of Wealth" as a decision criterion for single-period portfolio optimization based on the expected returns and standard deviations of the compared portfolios. It relies on a rational investor with risk aversion and a decision criterion based on indifference curves of risk/returns assumed to represent what may be an investor's quadratic utility.

Roy's safety-first criterion, starting in 1952 with Arthur Roy's paper "Safety First and the Holding of Assets," a portfolio selection criterion based on the comparison of the probabilities of expected returns falling below a minimum threshold. This threshold reflects the centrality of the goals of the empirical decision-maker in portfolio optimization.

Bounded Rationality, starting in 1957 with Herbert Simon's book *Models of Man* as an alternative to the quantitative modeling of rational decision-making, recognizing the limits of rationality when individuals make real-life decisions, thus becoming "satisficers" instead of maximizers. This formalizes the articulation of the differences between the theoretical observer and the empirical decision-maker.

Stochastic dynamic programming, starting in 1957 with Richard E. Bellman's book *Dynamic Programming,* as an optimal control modeling technique for decision-making under uncertainty. It uses recursive methods applied to an objective function, control variables, and state variables to find the optimizing policy function. Its value in real life is limited by the problem of dimensionality.

Hyperbolic discounted utility, an animal psychology finding about intertemporal preference reversal, starting in 1961 with Richard J. Herrnstein's paper "Matching Law," followed-up by George Ainslie in his 1974 paper "Impulse Control in Pigeons," and in 1992 with his book *Picoeconomics*.

Seven states of randomness, starting in 1964 with Benoit Mandelbrot's presentation to the Internal Congress for Logic "The Epistemology of Chance in Certain Newer Sciences," and followed-up in his 1997 book *Fractals and Scaling in Finance*. This work showed the existence of material model risk in financial theories, such as MPT, when they are exposed to the wilder forms of randomness. These findings were expanded by Nassim Nicholas Taleb with his books, including *Fooled by Randomness* in 2002 and *Statistical Consequences of Fat Tails* in 2020. This is the beginning of a change in perspective from seeing (mild) randomness as a measurement error in deterministic models, to seeing (wild) randomness as a structural feature of the stochastic nature of reality.

The efficient market hypothesis (EMH), starting in 1964 with Eugene Fama's PhD thesis, published in 1965, and formalized in 1970 by his paper "Efficient Capital Markets: A Review of Theory and Empirical Work." EMH asserts that asset prices reflect all available information and that prices only react to new information.

The Capital Asset Pricing Model (CAPM), starting in 1964 with William Sharpe's paper "Capital Asset Prices – A Theory of Market Equilibrium Under Conditions of Risk," as a model to price a theoretical rate of return for risky assets, based on one un-diversifiable, systematic risk factor, "beta," that is measured through historical regression analysis in order to build portfolios. This was developed as a method to simplify Markowitz's portfolio construction. However, betas may only explain 70% of portfolio returns.

Measuring Financial Risk Tolerance, starting in 1964 with N. Kogan and M.A. Wallach's book *Risk Taking: A Study in Cognition and Personality*, and followed up by K.R. MacCrimmon and D.A. Wehrung with their 1984 paper "The Risk In-Basket." This work is generally based on classic (psychometric) test theory (CTT). It was institutionalized into practice with standards such as the 2006 definition by the Organization for Standardization as follows: Risk tolerance is ["the extent to which someone is willing to experience a less favorable outcome in the pursuit of an outcome with more favorable attributes"]. However, a person's risk tolerance does not seem to be stable over time or under different circumstances.

Intertemporal choice in life-cycle finance, starting in 1966 with Franco Modigliani's book *The Life Cycle Hypothesis of Savings, the Demand for Wealth and the Supply of Capital* and extended in 2002 with Zvi Bodie's paper "Life-Cycle Finance in Theory and in Practice," as a process for intertemporal choice and trade-offs between (i) work, consumption, savings, borrowing, etc., with (ii) a premise of a preference for consumption smoothing, (iii) subject to the maximization of a utility function, resulting in (iv) wealth allocations across a range of risk management techniques that include hedging and insuring in addition to diversified market exposures.

Intertemporal portfolio choice, starting with Robert Cox Merton in 1969 with his paper titled "Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case," as a decision process to optimally allocate wealth to financial assets over time. Merton derived a closed-form optimal allocation formula, Merton's fraction, in the case of continuous time for an investor with hyperbolic absolute risk aversion, with independently and identically distributed (IID) asset returns that follow a Brownian motion.

The law of proportionality of risk and return, formalized by Robert Cox Merton's 1972 paper "An analytic derivation of the efficient portfolio frontier," such that risk and return can be engineered by mixing the weight of the expected value of risky assets with the weight of the risk-free asset.

The Treynor-Black model for security selection, starting in 1973 with the paper by Jack Treynor and Fisher Black "How to use Security Analysis to Improve Portfolio Selection," such that the optimal two-part portfolio has a passive, price-weighted index portfolio and an active portfolio based on predictions about security alpha.

The Black-Scholes equation for hedging options, starting with the 1973 paper by Fisher Sheffey Black and Myron Samuel Scholes "The Pricing of Options and Corporate Liabilities," as a hedging formula for buying the underlying assets to eliminate the risk of holding the matching European call or put options, under the assumption of price changes following a geometric Brownian motion in a frictionless market. The purpose of the formula is to estimate risk-neutral option pricing with dynamic hedging.

Arbitrage pricing theory (APT), starting in 1976 with Stephen Alan "Steve" Ross' paper "The arbitrage theory of capital asset pricing," an asset pricing model, proposed as an alternative for the capital pricing model of Sharpe, Lintner and Treynor. It is based on a weighted sum of historically regressed economic factors. This greater flexibility tends to over-fit the data.

Consumption-based CAPM (CCAPM), starting in 1978 with Robert Lucas in "Asset Prices in an Exchange Economy," a generalization of the single-period CAPM to the multi-period case in pricing the variance of potential consumption for asset holders.

Prospect theory, starting in 1979 with Kahneman and Tversky's paper "Prospect Theory: An Analysis of Decision under Risk," to explain the documented effects that violate the independence of the irrelevant alternatives (IIA) axiom of expected utility theory. Findings include: (i) risk aversion [a preference for smaller variance in outcomes**], (ii) loss aversion, and (iii) the endowment effect, [**the framing of decisions as gains or losses**]. Prospect theory changed the shape of the utility curve from concave functions over positive values to s-curve functions over positive and negative values.**

Universal portfolio management, starting in 1980 with the paper by Robert Bell and Thomas Cover "Competitive Optimality of Logarithmic Investment," and formalized in 1991 with Thomas M. Cover's paper "Universal Portfolios." This is an algorithmic approach to portfolio rebalancing. It learns adaptively from past data and maximizes the log-optimal growth rate.

Stochastic portfolio theory, starting in 1982 with E. Robert Fernholz's paper "Stochastic Portfolio Theory and Stochastic Market Equilibrium," followed by his book

***Stochastic Portfolio Theory* in 2002. This is a descriptive theory that uses log returns to remove the upward bias inherent in arithmetic returns.**

Asset allocation, starting in 1986 with the paper by Gary P. Brinson, L. Randolph Hood, and Gilbert L. Beebower "Determinants of Portfolio Performance," seeking to quantify "… which investment decisions had the greatest impact on the magnitude of total returns" but providing an answer of a 94.6% impact on the variance of total returns. The paper by Xiong, Ibbotson, Idzorek, and Chen (2010) "The Equal Importance of Asset Allocation and Active Management" shows that asset allocation may account for 12.5% of total return variation. Asset allocation may be a confounded variable with 75%+ of the effect coming from market exposure.

Cumulative prospect theory, starting in 1992 by Tversky and Kahneman's paper "Advances in Prospect Theory: Cumulative Representation of Uncertainty" as an answer to the "four-fold pattern of preferences" objection to prospect theory. The result is a "probability-weighting function" based on the observation that decision-makers switch their approach to risk in situations with very low or very high probabilities.

The Black-Litterman model for portfolio construction, starting in 1992 with the paper by Fisher Black and Robert Litterman "Asset Allocation Combining Investors Views with Market Equilibrium," obviating the need to estimate mean-variance-correlation parameters for MPT optimization by setting initial parameters to reflect current market prices. Institutional users can then modify parameters as their opinions differs from the markets.

The Fama-French three-factor CAPM, starting in 1992 with the paper by Eugene Fama and Kenneth French, "The Cross-Section of Expected Stock Returns," combining (i) market risk, (ii) small versus large companies, and (iii) high versus small book/market ratios to explain 90% of portfolio returns, as contrasted with the original CAPM single market risk factor (beta) which may only explain 70% of portfolio returns. Extensions include the Carhart four-factor CAPM, starting in 1997 with Mark Carhart's paper "On Persistence in Mutual Funds Performance," and the Fama-French five-factor CAPM, starting in 2015 with the paper by Eugene Fama and Kenneth French, "Dissecting anomalies with a five-factor model."

Post-modern portfolio theory, a proprietary solution starting in 1993 with a paper by Brian Rom and Kathleen Ferguson, "Post-Modern Portfolio Theory Spawns Post-Modern Optimizer," recognizes the asymmetrical distribution of risk and proposes solutions to manage downside-risk, including the development of the Sortino ratio.

Growth-based institutional intertemporal surplus optimization, starting in 1993 with Robert Cox Merton's paper "Optimal investment strategies for university endowment funds," and synthesized in 2004 in a paper by Markus Rudolf and William Ziemba

"Intertemporal surplus management" in the form of a four-fund approach that includes: (i) the market portfolio, (ii) the hedge portfolio for the state variable, (iii) the hedge portfolio for the liabilities, and (iv) the riskless asset.

Behavioral utility, starting in 1999 with a paper by Ernst Fehr and Klaus M. Schmidt "A Theory of Fairness, Competition and Cooperation," seeking to reconcile homo economicus utility functions with social preferences for fairness and cooperation, resulting in increasingly complex utility functions.

The 21st Century

Timeline Era Commentary

Psychological explanations proved too unconstrained for practical solutions. Mathematics reasserts its role as a method of skepticism about the ergodicity of observables, and thus the proper use of expected values. Mathematics becomes a language to question the structure of economic problems instead of a language that defines the structure economics problems. This creates a growing catalog of solutions to the traditional list of empirical puzzles, paradoxes, and anomalies.

Behavioral portfolio theory, starting in 2000 with a paper by H. Sherfin and Meir Statman, "Behavioral Portfolio Theory," a descriptive theory related to Roy's safety-first decision criterion, and Lola Lopes' security-potential/aspiration theory seeking to improve cumulative prospect theory utility functions and expressed in the form of a three-fund pyramid representing three different portfolios, thus departing from MPT's one portfolio recommendation. In 2009, Philippe de Brouwer's paper, "Maslowian Portfolio Theory: An Alternative formulation of the Behavioral Portfolio Theory," provides the justification to expand the number of portfolios.

The adaptive market hypothesis, starting in 2002 with a paper by Andrew Lo and Archie Craig MacKinlay, "A non-random walk down Wall Street," refuting the random walk nature of stock prices and the EMH by showing that there is a measure of predictability in market price changes. In 2004 Andrew Lo's paper "Adaptive-market theory offers investor insights" proposes to reconcile EMH and behavioral economics, suggesting that loss aversion, overconfidence, overreaction, and other behavioral finance biases reflect individual adaptation to changing markets using simple heuristics.

Risk parity portfolio management, a decade-old practice formally named in 2005 in Edward Qian's paper "Risk parity portfolios: Efficient portfolios through true diversification," based on the observation that a 60/40/10 asset allocation is a 90/10 risk allocation.

Liability-driven investment strategies, starting in 2008 with Roy Hoevenaars' paper "Strategic Asset Allocation and Asset Liability Management," implementing

institutional asset/liability matching and portfolio asset dedication in the broader context of the household balance sheet.

<u>The embedding of randomness in time in addition to the ensemble</u>, starting in 2011 with Ole Peters' paper "The time resolution of the St. Petersburg paradox," and expanded in the 2018 and 2019 versions of the "Course Notes" on ergodicity economics, establishing the physical meaning of the means from the collective meaning of the ensemble average to the individual meaning of the time average. In many investment models, decision criteria as well as risk measures rely on the expected value, thus embedding risk in the ensemble. Such measures are only relevant for individual decisions if the observable is ergodic. However, systems that grow are non-ergodic, they have histories generated by their unique growth rates. The use of expected values for non-ergodic observables is the foundation of many puzzles, paradoxes, and anomalies that arise between theoretical models and empirical observations. The ergodicity problem is a blind spot, not just a gap, in the investment management literature.

<u>Growth optimal decision criterion</u>, starting in 2011 with Ole Peters' paper "The time resolution of the St. Petersburg paradox," and followed-up by the 2016 paper by Ole Peters & Murray Gell-Mann "Evaluating gambles using dynamics." This work anchors ergodicity economics in the processes of decision theory, developing the growth optimal criterion of the time average growth rate, or its ergodic ensemble average alternative when justifiably available.

<u>Growth optimal leverage</u>, starting in 2011 with Ole Peters' paper "Optimal leverage from non-ergodicity" establishing that one cannot improve time average growth rates through asset allocations that depart from optimal leverage: 100% of the growth optimal selection.

<u>Growth optimal equity premium</u>, starting in 2013 with a paper by Ole Peters and Alex Adamou, "Stochastic market efficiency." This work extends the paper on leverage optimization with a theory of noisy stock prices, thus providing a solution to the equity premium puzzle [The empirical observation that average real returns on equities are much higher than the average real returns on bonds**], as an endogenous level of noise related to the rate of growth.**

<u>Growth optimal insurance</u>, starting in 2015 with a paper by Ole Peters and Alex Adamou, "Rational insurance with linear utility and perfect information." This work contrasts the win-lose changes in expected wealth that come from entering into insurance contracts, with the presence of win-win increases in time-average growth rates for both parties and over a range of prices.

<u>Irreproducible research results due to non-ergodicity</u>, starting in 2017 with a paper by Ole Peters and Maximilian J. Werner, "A recipe for irreproducible results." This work

shows that the assumption of the existence of a true value must be double-checked with the existence of a scaling factor for the variance of the time average that converges to zero.

Growth optimal utility functions, starting in 2018 with a paper by Ole Peters and Alex Adamou, "The time interpretation of expected utility theory." This work shows that utility functions are not irrational psychological re-weightings of monetary accounts that are external to the model, but instead are non-linear transformations that are internal to the model and that define ergodic observables on non-ergodic processes. Working with ergodic observables validates the use of statistical tools including the (i) theory of stationary processes, (ii) limit theorems, and (iii) the law of large numbers.

Growth optimal preference reversals, starting in 2019 with a paper by Ole Peters et al. "Microfoundations of Discounting." This work provides an endogenous model explanation based on time average growth rates instead of an appeal to psychological explanations that are external to the model.

The value of cooperation, starting in 2019 with a paper by Ole Peters and Alex Adamou, "The evolutionary advantage of cooperation." This work shows that increasing the number of cooperators increases the time average growth rate to the limit of the ensemble average growth rate. This is another way to show the value of diversification.

Growth optimal probability weighting, starting in 2020 with a paper by Ole Peters et al., "What are we weighting for? A mechanistic model for probability weighting." This work provides a mechanistic explanation for the probability weighting solution of cumulative prospect theory: The solution hinges on the difference in estimates of expected values and variances between a disinterested observer (DO) and the decision maker (DM), thus obviating the need for psychological explanations.

Growth optimal risk preferences, starting in 2020 with a paper by Yonatan Berman and Mark Kirstein, "Risk Preferences in Time Lotteries." This work shows that expected discounted utility (EDU) is inferior to growth optimality as a decision-making criterion for choices facing uncertainty in the timing of rewards.

Growth optimal measures of wealth inequality, starting in 2016 with a paper by Yonatan Berman, Ole Peters, and Alex Adamou, "Far from equilibrium: Wealth reallocation in the United States," and a 2020 update "Wealth Inequality and the Ergodic Hypothesis: Evidence from the United States." This work models the magnitude and the trend of wealth reallocation between the large group of individuals and the winners-take-all in the economy.

Growth optimal measures of economic growth, suggested in 2016 at the end of a paper by Alexander Adamou and Ole Peters, "Dynamics of inequality," and further

developed in a 2020 paper by Alexander Adamou, Yonatan Berman, and Ole Peters, "The Two Growth Rates of the Economy." This work presents a time average alternative to the rate of relative change in growth domestic product (GDP) per capita. The difference between the two growth rates offers a new measure of income inequality.

Conclusion

The roots of (i) expected value optimization, (ii) client-centric planning, and (iii) growth optimal selection twist and turn around one another over a centuries-long research history. This paper traces each strand chronologically. It also highlights their interconnections. Its goal is to connect what practitioners already know with ergodicity economics because it is the next wave of product development and client advice.

We cannot individually experience ensemble averages, thus using expected value as the default investment decision criterion has created a large catalog of empirical puzzles, paradoxes, and anomalies. Financial economics uses mathematics as a language to define the structure of investment problems. Ergodicity economics restores mathematics as a method of skepticism to question the structure of economic and investment problems. This change has created a growing catalog of solutions to the traditional list of empirical puzzles, paradoxes, and anomalies.

These connections between financial economics and ergodicity economics exist at a high level as shown in the timeline eras commentaries:

- The 19th century and earlier: *It is remarkable that the distinction between the arithmetic mean and the geometric mean starts with Euclid. This distinction is the oldest conceptual entry in the chronology, and it is also one of the key differences between financial economics and ergodicity economics. Another critical feature appears early on. The importance of the logarithm function is discovered in the 18th century. Further, and based on an effort to simplify complicated calculations in statistical mechanics, the concept of ergodicity emerges in the 29th century. Finally, even though both mathematical means appeared at the beginning of this chronology, we observe that expected value dominated the geometric mean early on, perhaps on account of its simpler mathematics. As the 20th century starts, the stage is set for continued work on understanding the meaning of the means based on the microstructure of the data. However, it will take another 99 years before the meaning of the means becomes clear enough to offer generalized solutions beyond the expected value.*

- The 20th century through World War II: *The first half of the 20th century axiomatized the mathematics of expected value optimization, introducing simplifying modeling assumptions. This period also introduced the first client-*

centric entries in this chronology. The difference and tension between the theoretical observer and the empirical decision-maker remained unresolved without an appeal to unconstrained psychological reasons, and until the advent of ergodicity economics.

- The 20th century, post-World War II: *The second-half of the 20th century started with the formalization of mathematics as a language that defines the structure of economic problems. In particular, asserting the ergodicity of economic observables enshrined expected value as the decision criterion. This created a large catalog of empirical puzzles, paradoxes, and anomalies that drove many researchers to look for solutions with unconstrained psychology explanations, before and instead of time average growth rate optimization.*

- The 21st century: *Psychological explanations proved too unconstrained for practical solutions. Mathematics reasserts its role as a method of skepticism about the ergodicity of observables, and thus the proper use of expected values. Mathematics becomes a language to question the structure of economic problems instead of a language that defines the structure economics problems. This creates a growing catalog of solutions to the traditional list of empirical puzzles, paradoxes, and anomalies.*

These connections also exist at the level of individual conceptual entries as shown with the visual differentiations between the three braids of the story: expected value, retirement planning, and growth optimality.

Finally, this narrated conceptual chronology proved to be a helpful exercise to connect the dots between financial economics and ergodicity economics. It also proved to be a productive platform for the development of additional content: Identifying and rank-ordering the empirical puzzles, paradoxes, and anomalies of financial economics that matter the most for practitioners and that could benefit from solutions based on ergodicity economics. This will be the topic of another paper.

Copyright © 2020

The Curve, Triangle & Rectangle Institute℠

Ergodicity Economics Papers – Reference List

Adamou, A., and O. Peters. 2016.

Dynamics of Inequality. *Significance* 13, no. 3 (June): 32–37.

Adamou, A., Y. Berman, D. Mavroyiannis, and O. Peters. 2019. Microfoundations of Discounting.

RESEARCHERS.ONE. https://www.researchers.one/article/2019-10-7.

Adamou, A., Y. Berman, and O. Peters. 2019. The Two Growth Rates of the Economy

RESEARCHERS.ONE. https://www.researchers.one/article/2020-09-22.

Berman, Y., O. Peters, and A. Adamou. 2016. (Updated 2017, 2019). An Empirical Test of the Ergodic Hypothesis: Wealth Distribution in the United States.

SSRN-id2794830.pdf

Berman, Y., O. Peters, and A. Adamou. 2016. Far from equilibrium: Wealth reallocation in the United States.

https://arxiv.org/abs/1605.05631v1

Berman, Y., O. Peters, and A. Adamou. 2020. Wealth Inequality and the Ergodic Hypothesis: Evidence from the United States.

RESEARCHERS.ONE. https://www.researchers.one/article/2020-07-15.

Berman, Y., and M. Kirstein. 2020. Risk Preferences in Time Lotteries.

RESEARCHERS.ONE. https://www.researchers.one/article/2020-02-16.

Kirstein, M. 2015. From the Ergodic Hypothesis in Physics to the Ergodic Axiom in Economics.

7th Winter Conference of Institute for the Comprehensive Analysis of the Economy, University of Linz, December 4–5, 2015.

Meder, D., F. Rabe, T. Morville, K. H. Madsen, M. T. Koudahl, Ray J. Dolan, H. R. Siebner, and O. J. Hulme. 2019. Ergodicity-breaking Reveals Time Optimal Economic Behavior in Humans.

https://arxiv.org/abs/1906.04652.

Peters, O. 2009. On Time and Risk.

Santa Fe Institute Bulletin 24, no. 1: 36-41. http://tuvalu.santafe.edu/~ole/Peters2009.pdf.

Peters, O. 2011a. Optimal Leverage from Non-ergodicity.

Quantitative Finance 11, no. 11 (November): 1,593–1,602.

Peters, O. 2011b. The Time Resolution of the St. Petersburg Paradox.

Philosophical Transactions of the Royal Society A 369, no. 1956 (December): 4,913–4,931.

Peters, O. 2011c. Menger 1934 Revisited.

https://arxiv.org/abs/1110.1578.

Peters, O. 2019. Comment on D. Bernoulli (1738).

RESEARCHERS.ONE. https://www.researchers.one/article/2019-05-1.

Peters, O. 2019. The Ergodicity Problem in Economics.

Nature Physics 15, no. 12 (December): 1,216–1,221. https://www.nature.com/articles/s41567-019-0732-0.

Peters, O., A. Adamou. 2020. Ergodicity Economics Lecture Notes.

https://github.com/LMLhub/ee_lecture_notes. (removed by authors)

Peters, O., and A. Adamou. 2011. Leverage Efficiency.

https://arxiv.org/abs/1101.4548.

Peters, O., and A. Adamou. 2013. Stochastic Market Efficiency.

Working paper 06-022, Santa Fe Institute. https://sfi-edu.s3.amazonaws.com/sfi-edu/production/uploads/sfi-com/dev/uploads/filer/fd/b3/fdb3214b-5c1f-4832-8337-310dab0836b4/13-06-022.pdf.

Peters, O., and A. Adamou. 2015. Rational insurance with linear utility and perfect information

Peters, O., and A. Adamou. 2017. Insurance Makes Wealth Grow Faster.

https://arxiv.org/abs/1507.04655v2

Peters, O., and A. Adamou. 2017. Leverage Efficiency.

https://arxiv.org/abs/1101.4548v2

Peters, O., and A. Adamou. 2020. Leverage Efficiency.

RESEARCHERS.ONE. https://www.researchers.one/article/2020-06-11.

Peters, O., and A. Adamou. 2018. See also 2019 version on RESEARCHERS.ONE. An Evolutionary Advantage to Cooperation.

RESEARCHERS.ONE https://arxiv.org/pdf/1506.03414v2.pdf.

Peters, O., and A. Adamou. 2018. Ergodicity Economics Lecture Notes.

https://ergodicityeconomics.files.wordpress.com/2018/06/ergodicity_economics.pdf.

Peters, O., and A. Adamou. 2019. An Evolutionary Advantage of Cooperation.

RESEARCHERS.ONE. https://www.researchers.one/article/2019-03-04.

Peters, O., and A. Adamou. 2018. The Evolutionary Advantage of Cooperation. *(Updated from 2015).*

https://arxiv.org/pdf/1506.03414v2.pdf.

Peters, O., and A. Adamou. 2018. The Sum of Log-Normal Variates in Geometric Brownian Motion.

https://arxiv.org/abs/1802.02939.

Peters, O., and A. Adamou. 2018. The Time Interpretation of Expected Utility Theory.

https://arxiv.org/abs/1801.03680.

Peters, O., and M. Gell-Mann. 2016. Evaluating Gambles Using Dynamics.

Chaos: An Interdisciplinary Journal of Nonlinear Science 26 023103. https://aip.scitation.org/doi/10.1063/1.4940236.

Peters, O., and M. Gell-Mann. 2015. Evaluating Gambles Using Dynamics.

https://arxiv.org/abs/1405.0585V2.

Peters, O., and W. Klein. 2013. Ergodicity Breaking in Geometric Brownian Motion.

Physical Review Letters 100, 100603. http://tuvalu.santafe.edu/~ole/PetersKlein2013.pdf.

Peters, O., and M. Werner. 2017. A Recipe for Irreproducible Results.

https://arxiv.org/abs/1706.07773.

Peters, O., A. Adamou, M. Kirstein, and Y. Berman. 2020. Probability Weighting as Model Calibration.

RESEARCHERS.ONE. https://www.researchers.one/article/2019-03-04. (See also 2020-03-06)

About the Author

François Gadenne is a serial fintech entrepreneur with 40 years of experience. He emigrated from France to the United States in his early twenties after completing a Baccalaureate in Mathematics, Latin, and Greek, and receiving the diploma from the Ecole Superieure the Commerce de Paris. He also completed an MBA from the J. L. Kellogg Graduate School of Management at Northwestern University. In the 2000s and for nearly a decade, he was a lecturer at Boston University, teaching FI728 “Principles of Finance,” Professor Zvi Bodie’s class in the Master of Science in Investment Management (MSIM) program.

François’ current venture, the Curve, Triangle & Rectangle Institute℠ (CTRI℠), is a not-for-profit, membership-based research & development institute that was founded in 2018. CTRI extends the research findings of a previous venture, the Retirement Income Industry Association (RIIA), that are documented in the published paper "The Shapes of Retirement: Are you a Curve, a Triangle, or a Rectangle?" (Collins/Gadenne 2017). In 2017, the Investments & Wealth Institute® (formerly IMCA) acquired RIIA’s educational assets, including the Retirement Management Advisor® designation (RMA®), and the peer-reviewed *Retirement Management Journal* (*RMJ*).

CTRI explores the unseen connections in research, and translates research findings in plain English in order to (i) find the powerful ideas that are not widely known, (ii) flag the false positives, and (iii) validate the insights that can be transformed into differentiated business practices, or new ventures.

Membership is by invitation only, exclusive by industry, and includes leaders from an investment company, a life company, a financial distributor, a professional education provider, and a fintech start-up. Invitations have been extended to a Bank, and an RIA based on the personal recommendations from existing members. Members prioritize research based on the issues that keep them up at night. Findings are shared with members in membership notes, personal calls, and group meetings. Members have the option to customize the development of shared findings, under NDA.

From time to time, members approve a limited number of these findings for general publication. This paper was written for CTRI members who decided to share its content with the public based on the importance of the topic for the financial industry and its clients. François can be reached at fg@ctri-usa.org.

This paper benefited from detailed comments from Patrick Collins, John Dundas, Tim Garrett, Dan Keady, Simon Locke, Al Turco, William Selden, and Travis Smith.