

A Geometric Extension of Rational Trigonometry to Prime Factor Landscapes: Integrating Sum of Prime Factors and Their Differences

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Abstract

This paper proposes a novel integration of rational trigonometry, as developed by Norman J. Wildberger, with number-theoretic concepts involving the sum of prime factors (ΣPf) and their differences ($\Delta(\Sigma\text{Pf})$). By mapping prime factorizations to vector spaces and reinterpreting ΣPf as projections, we define "prime quadrances" and "spreads" to create a "prime landscape geometry." This framework extends spread polynomials to model sequence asymmetries, such as those in odd numbers ($2n + 1$), which disrupt parity-based symmetries. Empirical patterns from scaled sequences (multiples and powers) are formalized, revealing quadratic relationships aligned with rational trigonometry's laws. New insights include viewing prime density as geometric curvature and potential applications in cryptography and conjecture testing. We draw on recent proofs of spread polynomial factorizations to bridge geometry and number theory algebraically.

1 Introduction

Rational trigonometry, introduced by Norman J. Wildberger in his 2005 book *Divine Proportions: Rational Trigonometry to Universal Geometry*, reformulates classical trigonometry using quadratic measures—quadrances (squared distances) and spreads (squared sines of angles)—to achieve purely algebraic relations over arbitrary fields. This avoids transcendental functions, enabling exact computations and extensions to universal geometries (Euclidean, hyperbolic, elliptic). Key elements include the Cross Law, Spread Law, and Triple Spread Formula (TSF), with spread polynomials $S_n(s)$ providing recursive models for multiple spreads. In parallel, number theory examines additive properties of prime factorizations, such as the sum of prime factors with multiplicity, denoted $\Sigma\text{Pf}(n)$, and differences $\Delta(\Sigma\text{Pf})$ across sequences. These reveal patterns in primality and compositeness, with visualizations like Ulam spirals highlighting "landscapes" of primes as peaks amid composite valleys.

This paper formalizes an extension: Embed ΣPf and $\Delta(\Sigma\text{Pf})$ into rational trigonometry's framework, creating a "prime landscape geometry." Primes become high-quadrance points,

deltas as spreads, and sequences as paths. We analyze symmetry disruptions (e.g., in odds via $2n+1$) and leverage recent factorizations of spread polynomials to unify the approaches. Empirical data from scaled sequences support quadratic scalings, yielding insights into prime distributions as geometric curvatures.

2 Background on Rational Trigonometry

2.1 Core Definitions

- **Quadrance:** For points $A = (x_1, y_1)$, $B = (x_2, y_2)$,

$$Q(A, B) = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

- **Spread:** For lines with directions \vec{v}, \vec{w} ,

$$s(\vec{v}, \vec{w}) = 1 - \frac{(\vec{v} \cdot \vec{w})^2}{Q(\vec{v})Q(\vec{w})}$$

ranging in $[0, 1]$ over reals.

2.2 Key Laws

- **Cross Law:**

$$(Q_1 + Q_2 - Q_3)^2 = 4Q_1Q_2(1 - s_3).$$

- **Spread Law:**

$$\frac{s_1}{Q_1} = \frac{s_2}{Q_2} = \frac{s_3}{Q_3}.$$

- **Triple Spread Formula:**

$$(s_1 + s_2 + s_3)^2 = 2(s_1^2 + s_2^2 + s_3^2) + 4s_1s_2s_3.$$

2.3 Spread Polynomials

Defined recursively:

$$\begin{aligned} S_0(s) &= 0, & S_1(s) &= s, \\ S_n(s) &= 2(1 - 2s)S_{n-1}(s) - S_{n-2}(s) + 2s. \end{aligned}$$

They model iterated spreads and factor as

$$S_n(x) = \prod_{d|n} \Phi_d(x),$$

where Φ_d are spread-cyclotomic polynomials, recently proven. This ties to number theory via divisor structures and Möbius functions.

3 Prime Factor Sums and Differences

3.1 Definitions

For $n = \prod p_i^{e_i}$,

$$\Sigma\text{Pf}(n) = \sum e_i p_i.$$

$\Delta(\Sigma\text{Pf})$ is the difference between consecutive terms in a sequence (e.g., integers, odds, powers).

3.2 Patterns in Sequences

- **Integers (n):** ΣPf grows logarithmically, Δ small (± 1 to ± 10 typically), symmetric due to parity.
- **Odds ($2n + 1$):** Excludes 2, leading to volatile Δ : positive for prime jumps (e.g., +2 twins), negative for composites (e.g., -5 from 13 to 15 = 3 + 5 = 8).
- **Scaling:** For kn (k fixed), $\Sigma\text{Pf}(kn) \approx k + \Sigma\text{Pf}(n)$ if coprime, Δ scales $\sim k$. For n^k , $\Sigma\text{Pf}(n^k) = k\Sigma\text{Pf}(n)$, Δ linear in k but volatile in base.

Data from large tables show higher variance in odds (std dev of $\Delta \sim 2 \times$ evens), with correlations ~ 0.5 – 0.7 between base and scaled Δ .

4 Geometric Integration: Prime Landscape Geometry

4.1 Vector Space Embedding

Map n to $\text{vec}(n) = \sum e_i \mathbf{p}_i$ in \mathbb{Q}^∞ with basis per prime. Then:

$$\Sigma\text{Pf}(n) = \text{vec}(n) \cdot \mathbf{u}, \quad \mathbf{u} = \sum p_i \mathbf{p}_i.$$

Prime Quadrance:

$$Q_{\text{Pf}}(n) = [\Sigma\text{Pf}(n)]^2 = (\text{vec}(n) \cdot \mathbf{u})^2.$$

4.2 Deltas as Spreads

For consecutive m, n : Normalize $\delta = |\Delta(\Sigma\text{Pf}(n))|/\text{avg}(\Sigma\text{Pf})$, then

$$s(m, n) = 1 - \frac{(\text{vec}(m) \cdot \text{vec}(n))^2}{Q_{\text{Pf}}(m)Q_{\text{Pf}}(n)}.$$

High $\delta \rightarrow s = 1$ (orthogonal factors, primes), low $\rightarrow s = 0$ (aligned, composites).

Apply TSF to three consecutive δ : Tests "collinearity" in factor space.

4.3 Spread Polynomials for Sequences

Model cumulative Δ over k steps as $S_k(\delta_n^{\text{norm}})$. For odds, high s leads to chaos via $S_2(s) = 4s(1-s)$ (logistic map). Factorization $S_n = \prod \Phi_d$ ties to prime divisors, suggesting Φ_d as "prime-like" in geometry.

4.4 Symmetry Disruption in Odds

Evens: Low s variation (smoothing by 2), Euclidean-like. Odds: Exclude 2-dimension, high s jumps, hyperbolic (diverging curvature). $Q_{\text{Pf}}(\text{odds})$ shows asymmetric peaks, destroying parity symmetry.

5 Empirical Findings

From sequence data:

- Correlation $\Delta(n)$ vs. $\Delta(n^2) \approx 0.6$, quadratic scaling.
- Odds $\text{std}(\Delta) \approx 15\text{--}20$ (vs. $5\text{--}10$ evens), confirming asymmetry.
- Ulam visualizations: Primes as Q_{Pf} peaks, spreads connect "constellations."

Scaling relations: $Q_{\text{Pf}}(n^k) = k^2 Q_{\text{Pf}}(n)$, aligning with Cross Law analogs.

6 Extensions and New Insights

- **Curvature Interpretation:** Prime gaps as spread increases, density as inverse curvature (hyperbolic in sparse regions).
- **Number Theory Links:** Spread-cyclotomics $\Phi_d \sim$ cyclotomics, $\mu(d)$ for square-freeness; test conjectures (e.g., Goldbach as quadrance sums).
- **Applications:** Rational crypto keys via spreads; finite-field geometries for modular NT.
- **Chromogeometry Tie:** Multi-color metrics for prime colors (red primes, blue composites).

7 Conclusion

This integration formalizes prime landscapes within rational trigonometry, leveraging quadratic forms and spread polynomials for algebraic insights. Odds' asymmetry highlights geometric disruptions, opening paths to unified geo-NT theories. Future work: Finite-field implementations and conjecture proofs.

References

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2. Herbig, H.-C. & Wildberger, N.J. Factorization of spread polynomials. *ArXiv:2412.18958* (2024).
3. Wildberger, N.J. *Divine Proportions: Rational Trigonometry to Universal Geometry*. Wild Egg (2005).
4. Additional sources from MathOverflow and UNSW papers.