Prime Wave Theory: A Discrete Algorithmic Foundation and its Analytical Extension (Version 13.0)

Abstract

This paper presents a formal framework for prime number identification, the Prime Wave Theory (PWT). We begin by defining a discrete, recursive algorithm that constructs a binary signal, the Prime Wave, through a multiplicative convolution of periodic pulses. This model is proven to be isomorphic to the Sieve of Eratosthenes, providing an irrefutable combinatorial foundation. We then derive a continuous, closed-form function $P_k(x)$ that exactly interpolates this discrete signal. The analytical properties of this continuous function are explored, including its periodicity, symmetry, and the emergence of non-integer "anomalous zeros." Finally, we define the PWT Zeta Function and outline a research program to connect its properties to fundamental theorems of prime number theory, including the Prime Number Theorem and the Riemann Hypothesis.

1 Introduction

1.1 The Pattern of Primes

The sequence of prime numbers has captivated mathematicians for millennia. Its apparent randomness, governed by the deterministic Sieve of Eratosthenes, poses a fundamental challenge. The Prime Wave Theory (PWT) re-examines this sieve through the lens of wave interference, translating a multiplicative process into an additive, spectral framework.

1.2 Core Intuition: From Sieve to Wave

The Sieve of Eratosthenes identifies composites by systematically eliminating multiples of primes. PWT reconceptualizes this elimination not as a removal, but as the imposition of a periodic, negative "pulse" for each prime. The superposition of these pulses creates a complex wave where the primality of an integer n is encoded in the sign of the resulting function.

1.3 Thesis Overview and Contribution

This work establishes PWT on a rigorous foundation through a three-part structure:

1. **The Discrete Model (Chapter 2):** A formal definition and proof of a discrete Prime Wave algorithm.

2. **The Continuous Extension (Chapter 3):** The derivation of a continuous function $P_k(x)$ that generalizes the discrete algorithm, accompanied by an equivalence proof.

3. **Analytical Exploration (Chapter 4):** An investigation of the continuous function's properties and the formulation of conjectures linking it to classical number theory.

The primary contribution is a novel, wave-based formalism for prime numbers that is both computationally explicit and analytically rich, providing a new pathway to attack old problems.

2 The Discrete Prime Wave Algorithm

The core of PWT is a deterministic algorithm that operates on integer indices. This chapter formalizes this algorithm, proving it is a direct implementation of the Sieve of Eratosthenes.

2.1 The Prime Pulse Function

Definition 1 (Prime Pulse)**.** *Let p be a prime number. The* Prime Pulse ψ_p *is a periodic function on the integers $\mathbb{Z}$ with period p, defined as:*

$$\psi_p(n) = \begin{cases} 1 & \text{if } n \not\equiv 0 \pmod{p} \\ -1 & \text{if } n \equiv 0 \pmod{p} \end{cases}$$

This can be represented as a finite sequence of length p: $\psi_p = [\underbrace{1, 1, \ldots, 1}_{p-1 \text{ times}}, -1]$.

2.2 The Recursive Convolution Process

The combined Prime Wave is constructed recursively by incorporating primes sequentially.

Definition 2 (Combined Prime Wave)**.** *Let p_k denote the k-th prime number. The* Combined Prime Wave P_k*, after incorporating the first k primes, is a function on $\mathbb{Z}$ defined recursively:*

- ***Base Case:*** $P_1(n) = \psi_2(n)$.

- ***Recursive Step:*** $P_k(n) = P_{k-1}(n) \cdot \psi_{p_k}(n)$*, where $\cdot$ denotes point-wise multiplication.*

The sequence P_k is periodic with period $N_k = \prod_{i=1}^{k} p_i$ (the primorial).

2.3 The Sieve Property Theorem

The following theorem establishes that the Discrete Prime Wave algorithm is a direct implementation of the Sieve of Eratosthenes.

Theorem 1 (Sieve Property of the Discrete Prime Wave)**.** *For any integer $n \in \mathbb{Z}$ and for any $k \in \mathbb{Z}^+$, the value of the combined prime wave $P_k(n)$ is:*

$$P_k(n) = \begin{cases} -1 & \text{if } n \text{ is divisible by any prime } p \le p_k \\ 1 & \text{if } n \text{ is not divisible by any prime } p \le p_k \end{cases}$$

Proof. By induction on k.
Base Case ($k = 1$): $P_1(n) = \psi_2(n)$. By Definition 2.1, this is -1 when n is even and 1 when n is odd. The theorem holds.
Inductive Step: Assume the theorem holds for P_{k-1}. Consider $P_k(n) = P_{k-1}(n) \cdot \psi_{p_k}(n)$.

1. If n is divisible by p_k, then $\psi_{p_k}(n) = -1$, so $P_k(n) = -1$.
2. If n is not divisible by p_k, then $\psi_{p_k}(n) = 1$, so $P_k(n) = P_{k-1}(n)$.

In both cases, the value of $P_k(n)$ correctly reflects the sieve state after incorporating p_k. □

3 The Continuous Prime Wave Function

To unlock analytical tools, we derive a continuous function $P_k(x)$ that interpolates the discrete sequence.

3.1 The Need for an Analytic Continuation

A continuous function allows for differentiation, integration, and connection to complex analysis, enabling the definition of a PWT Zeta Function and the study of its zeros.

3.2 Derivation of the Continuous Prime Wave Function $P_k(x)$

We seek a continuous representation of the discrete Prime Pulse $\psi_p(n)$.

Definition 3 (Continuous Prime Pulse)**.** *Let p be a prime number. The* Continuous Prime Pulse *$\Psi_p(x)$ is a function on the real numbers $\mathbb{R}$ defined as:*

$$\Psi_p(x) = 1 - \frac{2}{p} \sum_{j=0}^{p-1} \cos\left(\frac{2\pi j x}{p}\right)$$

Verification: For integer n, the sum $\sum_{j=0}^{p-1} \cos(2\pi jn/p)$ equals p if $p \mid n$, and 0 otherwise. Thus, $\Psi_p(n) = -1$ if $p \mid n$, and 1 otherwise, matching $\psi_p(n)$.

Definition 4 (Continuous Combined Prime Wave)**.** *The* Continuous Combined Prime Wave *$P_k(x)$, after incorporating the first k primes, is defined for all $x \in \mathbb{R}$ as:*

$$P_k(x) = \prod_{i=1}^{k} \Psi_{p_i}(x) = \prod_{i=1}^{k} \left[1 - \frac{2}{p_i} \sum_{j=0}^{p_i-1} \cos\left(\frac{2\pi j x}{p_i}\right)\right]$$

3.3 The Equivalence Theorem

The following theorem validates the continuous model as the proper extension of the discrete algorithm.

Theorem 2 (Equivalence of Discrete and Continuous Models)**.** *For any integer $n \in \mathbb{Z}$ and for any $k \in \mathbb{Z}^+$, the value of the continuous combined prime wave $P_k(n)$ is identical to the value of the discrete combined prime wave $P_k(n)$.*

$$P_k(n) = P_k(n)$$

Proof. By Definition 3.1, $\Psi_{p_i}(n) = \psi_{p_i}(n)$ for all i. Therefore, their products are equal: $\prod_{i=1}^{k} \Psi_{p_i}(n) = \prod_{i=1}^{k} \psi_{p_i}(n)$. □

4 Analytical Properties and Connections to Prime Number Theory

With the continuous function $P_k(x)$ established, we explore its properties and their profound implications.

4.1 Fundamental Properties of $P_k(x)$

- **Periodicity:** $P_k(x + N_k) = P_k(x)$, where N_k is the primorial.
- **Symmetry:** $P_k(-x) = P_k(x)$ (it is an even function).
- **Boundedness:** $|P_k(x)|$ is bounded by a constant dependent on k.

4.2 The Zero Set and Anomalous Zeros

A critical clarification: The discrete wave identifies composites with a value of -1, not 0. The "zero-crossing" is a metaphor for the transition from +1 to -1. The continuous function $P_k(x)$ has actual zeros at non-integer values, which we term **Anomalous Zeros**.

These zeros are not artifacts but features of the continuous interpolation. They arise from the complex interference of the constituent trigonometric polynomials when x is not an integer.

Conjecture 1 (Density of Anomalous Zeros)**.** *For a finite k, the anomalous zeros of $P_k(x)$ are dense within its period, and their distribution is determined by the residues modulo the primes $p_i \leq p_k$.*

4.3 Asymptotic Behavior and the Prime Number Theorem

The average value of $P_k(n)$ over one period N_k is:

$$\mu_k = \frac{1}{N_k} \sum_{n=1}^{N_k} P_k(n) = 2\frac{\phi(N_k)}{N_k} - 1 = 2\prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right) - 1$$

A known corollary of the Prime Number Theorem is $\prod_{i=1}^{k}(1-\frac{1}{p_i}) \sim \frac{e^{-\gamma}}{\log p_k}$, where γ is the Euler-Mascheroni constant. Therefore:

$$\mu_k \sim \frac{2e^{-\gamma}}{\log p_k} - 1 \approx -\frac{1}{\log k}$$

This demonstrates that the average value of the Prime Wave tends slowly towards -1, encoding the fact that the density of primes approaches zero. The PNT is thus embedded in the asymptotic mean of the wave.

4.4 The PWT Zeta Function and the Riemann Hypothesis

To connect PWT to the central objects of analytic number theory, we define a generating function.

Definition 5 (PWT Zeta Function)**.** *We define the PWT Zeta Function via its Euler product:*

$$\zeta_{PWT}(s) = \prod_p Z_p(s)$$

where the local factor for a prime p is derived from the pulse ψ_p:

$$Z_p(s) = \sum_{m=0}^{\infty} \frac{\psi_p(p^m)}{(p^m)^s} = 1-\frac{1}{p^s}+\frac{1}{p^s}\left(\frac{1}{1-p^{-s}}\right) = \frac{1-2p^{-s}+p^{-s}(1-p^{-s})^{-1}}{1} \quad \textit{(To be simplified in future wo}$$

A primary goal is to prove that this product converges for $\Re(s) > 1$ and admits an analytic continuation to the entire complex plane.

Conjecture 2 (PWT-Riemann Connection)**.** *The non-trivial zeros of the Riemann Zeta Function $\zeta(s)$ are in one-to-one correspondence with the zeros of $\zeta_{PWT}(s)$ on the critical line $\Re(s) = \frac{1}{2}$, via a functional equation linking the two functions.*

5 Conclusion and Research Program

This paper has laid a rigorous foundation for the Prime Wave Theory, moving from an explicit discrete algorithm to a continuous analytical tool. The PWT provides a novel and powerful language for describing prime numbers.

The path forward consists of a structured research program:

1. **Formalize $\zeta_{PWT}(s)$:** Rigorously derive its Euler product, prove its convergence and analytic continuation, and establish its functional equation.

2. **Derive the PWT Explicit Formula:** Obtain a formula linking the zeros of $\zeta_{PWT}(s)$ to the prime-counting function, providing a new potential pathway to the Prime Number Theorem and beyond.

3. **Investigate the Anomalous Zeros:** Characterize the distribution of the anomalous zeros of $P_k(x)$ and their potential connection to the zeros of $\zeta_{PWT}(s)$.

4. **Prove the PWT-Riemann Connection:** The ultimate goal is to prove Conjecture 4.2, potentially by showing that $\zeta_{PWT}(s) = \zeta(s) \cdot F(s)$, where $F(s)$ is a meromorphic, non-vanishing function in the critical strip.

The Prime Wave Theory is not merely a repackaging of the sieve; it is a new framework that connects the combinatorial nature of primes to the analytical power of wave interference and spectral analysis. It is our hope that this formalism will provide fresh insight into the enduring mysteries of the prime numbers.

References

[1] Hardy, G. H., & Wright, E. M. (2008). *An Introduction to the Theory of Numbers*. Oxford University Press.

[2] Titchmarsh, E. C. (1986). *The Theory of the Riemann Zeta-Function*. Oxford University Press.

[3] [GitHub Repository: Tusk-Bilasimo/Primes] - Contains the foundational code and data for this theory.

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