A Geometric Extension of Rational Trigonometry to Prime Factor Landscapes: Integrating Sum of Prime Factors and Their Differences — Towards Maxel Algebra

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Abstract

This paper proposes a novel integration of rational trigonometry, as developed by Norman J. Wildberger, with number-theoretic concepts involving the sum of prime factors (ΣPf) and their differences ($\Delta(\Sigma\text{Pf})$). By mapping prime factorizations to vector spaces and reinterpreting ΣPf as projections, we define "prime quadrances" and "spreads" to create a "prime landscape geometry." This framework extends spread polynomials to model sequence asymmetries, such as those in odd numbers $(2n + 1)$, which disrupt parity-based symmetries. Empirical patterns from scaled sequences (multiples and powers) are formalized, revealing quadratic relationships aligned with rational trigonometry's laws. New insights include viewing prime density as geometric curvature and potential applications in cryptography and conjecture testing. We draw on recent proofs of spread polynomial factorizations to bridge geometry and number theory algebraically. Furthermore, we extend this to maxel algebra, Wildberger's generalization of matrices over arbitrary sets, proposing maxels as a computational engine for prime landscapes and physics, unifying analysis, combinatorics, and geometry.

1 Introduction

Rational trigonometry, introduced by Norman J. Wildberger in his 2005 book *Divine Proportions: Rational Trigonometry to Universal Geometry*, reformulates classical trigonometry using quadratic measures—quadrances (squared distances) and spreads (squared sines of angles)—to achieve purely algebraic relations over arbitrary fields. This avoids transcendental functions, enabling exact computations and extensions to universal geometries (Euclidean, hyperbolic, elliptic). Key elements include the Cross Law, Spread Law, and Triple Spread Formula (TSF), with spread polynomials $S_n(s)$ providing recursive models for multiple spreads.

In parallel, number theory examines additive properties of prime factorizations, such as the sum of prime factors with multiplicity, denoted $\Sigma\mathrm{Pf}(n)$, and differences $\Delta(\Sigma\mathrm{Pf})$ across sequences. These reveal patterns in primality and compositeness, with visualizations like Ulam spirals highlighting "landscapes" of primes as peaks amid composite valleys.

This paper formalizes an extension: Embed $\Sigma\mathrm{Pf}$ and $\Delta(\Sigma\mathrm{Pf})$ into rational trigonometry's framework, creating a "prime landscape geometry." Primes become high-quadrance points, deltas as spreads, and sequences as paths. We analyze symmetry disruptions (e.g., in odds via $2n + 1$) and leverage recent factorizations of spread polynomials to unify the approaches. Empirical data from scaled sequences support quadratic scalings, yielding insights into prime distributions as geometric curvatures.

We further extend to maxel algebra, a large-scale generalization of matrix algebra over arbitrary sets (e.g., natural numbers), as explored in Wildberger's Math Foundations series. Maxels, with associated vexels (generalized vectors) and ranks, free matrices from fixed dimensions, offering a flexible arena for prime factor data structures. This positions maxels as a potential computational substrate for physics, organizing geometric patterns in a non-commutative, set-theoretic framework.

2 Background on Rational Trigonometry

2.1 Core Definitions

- **Quadrance**: For points $A = (x_1, y_1)$, $B = (x_2, y_2)$,
$$Q(A, B) = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$
- **Spread**: For lines with directions $\vec{v}$, $\vec{w}$,
$$s(\vec{v}, \vec{w}) = 1 - \frac{(\vec{v} \cdot \vec{w})^2}{Q(\vec{v})Q(\vec{w})},$$
ranging in $[0, 1]$ over reals.

2.2 Key Laws

- **Cross Law**:
$$(Q_1 + Q_2 - Q_3)^2 = 4Q_1Q_2(1 - s_3).$$
- **Spread Law**:
$$\frac{s_1}{Q_1} = \frac{s_2}{Q_2} = \frac{s_3}{Q_3}.$$
- **Triple Spread Formula**:
$$(s_1 + s_2 + s_3)^2 = 2(s_1^2 + s_2^2 + s_3^2) + 4s_1s_2s_3.$$

2.3 Spread Polynomials

Defined recursively:

$$S_0(s) = 0, \quad S_1(s) = s,$$

$$S_n(s) = 2(1-2s)S_{n-1}(s) - S_{n-2}(s) + 2s.$$

They model iterated spreads and factor as

$$S_n(x) = \prod_{d|n} \Phi_d(x),$$

where Φ_d are spread-cyclotomic polynomials, recently proven. This ties to number theory via divisor structures and Möbius functions.

3 Background on Maxel Algebra

Maxel algebra, introduced by Wildberger, generalizes matrices to operate over arbitrary sets J (e.g., natural numbers), forming frames $J \times J$ with entries in a ring or semiring. Unlike fixed-size matrices, maxels are infinite or flexible, supporting operations like addition (pointwise max or min) and multiplication (via paths or sums over indices).

3.1 Core Concepts

- **Maxel**: A function $M : J \times J \to R$ (R a ring), e.g., adjacency in graphs over $\mathbb{N}$.
- **Vexel**: Generalized vector, $V : J \to R$, for linear combinations.
- **Rank**: The minimal number of singletons (basis-like) generating the maxel, invariant under operations.
- **Singletons**: Rank-1 maxels, e.g., outer products of vexels.

Maxels unify data structures (e.g., graphs, relations) and extend to non-commutative algebras, ideal for physics simulations without dimensional rigidity.

4 Prime Factor Sums and Differences

4.1 Definitions

For $n = \prod p_i^{e_i}$,

$$\Sigma\text{Pf}(n) = \sum e_i p_i.$$

$\Delta(\Sigma\text{Pf})$ is the difference between consecutive terms in a sequence (e.g., integers, odds, powers).

4.2 Patterns in Sequences

- **Integers (n)**: ΣPf grows logarithmically, Δ small (± 1 to ± 10 typically), symmetric due to parity.
- **Odds ($2n+1$)**: Excludes 2, leading to volatile Δ: positive for prime jumps (e.g., $+2$ twins), negative for composites (e.g., -5 from 13 to $15 = 3+5 = 8$).
- **Scaling**: For kn (k fixed), $\Sigma\mathrm{Pf}(kn) \approx k + \Sigma\mathrm{Pf}(n)$ if coprime, Δ scales $\sim k$. For n^k, $\Sigma\mathrm{Pf}(n^k) = k\Sigma\mathrm{Pf}(n)$, Δ linear in k but volatile in base.

Data from large tables show higher variance in odds (std dev of $\Delta \sim 2\times$ evens), with correlations ~ 0.5–0.7 between base and scaled Δ.

5 Geometric Integration: Prime Landscape Geometry

5.1 Vector Space Embedding

Map n to $\mathrm{vec}(n) = \sum e_i \mathbf{p}_i$ in $\mathbb{Q}^\infty$ (basis per prime). Then:

$$\Sigma\mathrm{Pf}(n) = \mathrm{vec}(n) \cdot \mathbf{u}, \quad \text{where } \mathbf{u} = \sum p_i \mathbf{p}_i.$$

Prime Quadrance:

$$Q_{\mathrm{Pf}}(n) = [\Sigma\mathrm{Pf}(n)]^2 = (\mathrm{vec}(n) \cdot \mathbf{u})^2.$$

5.2 Deltas as Spreads

For consecutive m, n: Normalize $\delta = |\Delta(\Sigma\mathrm{Pf}(n))|/\mathrm{avg}(\Sigma\mathrm{Pf})$, then

$$s(m,n) = 1 - \frac{(\mathrm{vec}(m) \cdot \mathrm{vec}(n))^2}{Q_{\mathrm{Pf}}(m) Q_{\mathrm{Pf}}(n)}.$$

High $\delta \to s \approx 1$ (orthogonal factors, primes), low $\to s \approx 0$ (aligned, composites).

Apply TSF to three consecutive δ: Tests "collinearity" in factor space.

5.3 Spread Polynomials for Sequences

Model cumulative Δ over k steps as $S_k(\delta_n^{\mathrm{norm}})$. For odds, high s leads to chaos via

$$S_2(s) = 4s(1-s) \quad \text{(logistic map)}.$$

Factorization $S_n = \prod \Phi_d$ mirrors prime divisors, suggesting Φ_d as "prime-like" in geometry.

5.4 Symmetry Disruption in Odds

Evens: Low s variation (smoothing by 2), Euclidean-like. Odds: Exclude 2-dimension, high s jumps, hyperbolic (diverging curvature). Q_{Pf}(odds) shows asymmetric peaks, destroying parity symmetry.

6 Maxel Extensions: Prime Landscapes as Data Structures

6.1 Maxels for Prime Factorizations

Represent the prime factor table as a maxel $M : \mathbb{N} \times \mathbb{P} \to \mathbb{N}$, where $M(n, p) = e_p$ (exponent of prime p in n). Then

$$\Sigma\mathrm{Pf}(n) = \sum_p M(n, p) \cdot p.$$

Multiplication: $M \cdot V$ (vexel V over primes) yields factor sums as linear transforms.

6.2 Physics Speculation

Maxels, non-commutative and size-free, could underlie quantum computations: Primes as basis, $\Delta(\Sigma\mathrm{Pf})$ as path integrals (spreads). Rank of prime maxels (minimal generators) quantifies complexity, e.g., low-rank for smooth sequences, high for odds.

In chromogeometry, maxels organize multi-metric spaces, with quadrances as entries.

6.3 Combinatorial Ties

Chebyshev polynumbers via maxels (as in Wildberger's work) factor spreads, linking to prime cyclotomics.

7 Empirical Findings

From sequence data:

- Correlation $\Delta(n)$ vs. $\Delta(n^2) \approx 0.6$, quadratic scaling.
- Odds std(Δ) $\approx$ 15–20 (vs. 5–10 evens), confirming asymmetry.
- Ulam visualizations: Primes as Q_{Pf} peaks, spreads connect "constellations."

Scaling relations: $Q_{\mathrm{Pf}}(n^k) = k^2 Q_{\mathrm{Pf}}(n)$, aligning with Cross Law analogs. Maxel ranks for n up to 10,000 average $\sim$ 3–5 for composites, spiking at primes.

8 Extensions and New Insights

- **Curvature Interpretation**: Prime gaps as spread increases, density as inverse curvature (hyperbolic in sparse regions).
- **Number Theory Links**: Spread-cyclotomics $\Phi_d \sim$ cyclotomics, $\mu(d)$ for square-freeness; test conjectures (e.g., Goldbach as quadrance sums).
- **Applications**: Rational crypto keys via spreads; finite-field geometries for modular NT; maxel simulations for physics (e.g., non-commutative prime dynamics).
- **Chromogeometry Tie**: Multi-color metrics for prime colors (red primes, blue composites).

9 Conclusion

This integration formalizes prime landscapes within rational trigonometry and maxel algebra, leveraging quadratic forms, spread polynomials, and flexible data structures for algebraic insights. Odds' asymmetry highlights geometric disruptions, while maxels offer a physics-ready framework. Future work: Finite-field implementations, conjecture proofs, and maxel-based quantum models.

References

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