Character-Theoretic Advances in Prime Wave Theory: Exploring the Proofs from Version 15

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Abstract

Prime Wave Theory (PWT) offers a fresh, Fourier-analytic lens on the ancient Sieve of Eratosthenes, transforming the multiplicative process of prime identification into an additive spectral framework. Developed by the mathematician known as Tusk, the theory constructs a "Prime Wave"—a periodic function that encodes primality through wave interference patterns. The latest iteration, Version 15.1, released on October 1, 2025, expands on this by incorporating explicit connections to Dirichlet character theory and enhanced analysis of twin primes. This article focuses on the proofs from the research program excerpt "Character-Theoretic PWT and Twin Prime Spectral Analysis" (Part I: Character Theory of the Prime Wave), describing their mathematical structure, derivations, and implications.

1 Introduction to Prime Wave Theory

The full thesis, available here, provides rigorous proofs across discrete and continuous models, Fourier analysis, function spaces, and convergence theory. A key addition in V15.1 is the character-theoretic perspective, which positions PWT within analytic number theory and links it to L-functions.

PWT's core idea reconceptualizes the sieve as periodic "pulses" for each prime, multiplied to create a wave where values at integers indicate primality (1 for coprime, 0 for composites). The continuous extension $P_k(x)$ interpolates this, enabling spectral analysis. Recent discussions on platforms like X (from @PrimeWaveTheory) highlight its potential ties to cosmology and dark matter, but here we delve into the character theory proofs.

2 Part I: Character Theory of the Prime Wave

The excerpt introduces a character decomposition of the Prime Wave $P_k(x)$, with period $N_k = \prod_{i=1}^{k} p_i$ (the primorial). This section derives Fourier coefficients using Dirichlet characters, establishes bounds, and connects to L-functions.

2.1 Complete Character Decomposition

Theorem 1 (Character Expansion of Fourier Coefficients)**.** *For the Prime Wave $P_k(x)$ with period $N_k = \prod_{i=1}^{k} p_i$, the Fourier coefficients admit the exact expansion:*

$$c_m^{(k)} = \frac{1}{N_k} \sum_{d|\gcd(m,N_k)} \mu(d) \sum_{\substack{\chi \bmod N_k/d \\ \chi \text{ primitive}}} \bar{\chi}(m/d)\tau(\chi)$$

where $\tau(\chi) = \sum_{a=1}^{N_k/d} \chi(a) e^{2\pi i a/(N_k/d)}$ is the Gauss sum.

Proof Description

The proof begins with the characteristic function of coprime integers:

$$P_k(n) = \nVdash[\gcd(n, N_k) = 1] = \sum_{d|\gcd(n,N_k)} \mu(d)$$

This is Möbius inversion applied to the indicator of coprimality. The Discrete Fourier Transform (DFT) is then:

$$c_m^{(k)} = \frac{1}{N_k} \sum_{n=0}^{N_k-1} P_k(n) e^{-2\pi i m n/N_k}$$

Using the Chinese Remainder Theorem, the ring $\mathbb{Z}/N_k\mathbb{Z}$ factors into products modulo each prime p_i, and characters decompose accordingly:

$$\chi(n) = \prod_{i=1}^{k} \chi_i(n \bmod p_i)$$

For primitive characters modulo N_k/d, the Gauss sum magnitude is $|\tau(\chi)| = \sqrt{N_k/d}$. The key orthogonality relation:

$$\sum_{n \bmod N_k} \chi(n) e^{-2\pi i m n/N_k} = \bar{\chi}(m)\tau(\chi)$$

substitutes into the DFT sum, yielding the double summation over divisors and primitive characters. This decomposition provides "additional structure" beyond Ramanujan sums, as noted in the example for $k = 3$.

The proof is rigorous, leveraging standard number-theoretic tools to bridge the wave's spectrum to character sums.

Corollary 1 (Explicit Bound via Characters)**.**

$$|c_m^{(k)}| \le \frac{1}{N_k} \sum_{d|\gcd(m,N_k)} \phi(N_k/d)\sqrt{N_k/d} = O\left(\frac{(\log\log N_k)^2}{\sqrt{N_k}}\right)$$

This bound improves on cruder estimates like $|c_m^{(k)}| \le \phi(N_k)/N_k \sim e^{-\gamma}/\log k$, reflecting the refined character sum.

2.2 Character Isolation and Prime Patterns

Definition 1 (Character-Isolated Prime Wave)**.** *For a Dirichlet character* χ mod N_k*,*

$$P_k^{(\chi)}(x) = \frac{1}{\phi(N_k)} \sum_{a=1}^{N_k} \chi(a) P_k(x-a)$$

Theorem 2 (Character Isolation)**.** *The character-isolated wave satisfies:*

1. $P_k^{(\chi)}(n) \neq 0 \implies n \equiv a \pmod{N_k}$ *for some* a *with* $\chi(a) \neq 0$
2. *Fourier spectrum:* $\hat{P}_k^{(\chi)}(m) = \chi(m) c_m^{(k)}$
3. *For principal character* χ_0*:* $P_k^{(\chi_0)} = P_k$

Proof Description

The definition is a convolution over coprime residues. Property 1 follows from the vanishing condition outside coprime shifts. Property 2 uses the Fourier transform of convolution:

$$\widehat{P_k^{(\chi)}}(m) = \hat{\chi}(m) \cdot \hat{P}_k(m) = \chi(m) c_m^{(k)}$$

Property 3 is direct from $\chi_0(a) = 1$ for coprime a. This isolates spectral components by character, enabling "prime pattern" analysis.

2.3 Connection to Dirichlet L-Functions

Theorem 3 (Mellin Transform and L-Functions)**.** *The Mellin transform on [0,* N_k*]:*

$$\mathcal{M}P_k = \int_0^{N_k} P_k(x) x^{s-1} dx = \sum_{\substack{n<N_k \\ \gcd(n,N_k)=1}} n^{s-1}$$

For $\Re(s) > 1$*,*

$$\mathcal{M}P_k = N_k^s \prod_{i=1}^{k} \left(1 - p_i^{-s}\right) \zeta(s) + O(N_k^{\Re(s)-1})$$

Proof Description

The transform sums over coprime n. Möbius inversion rearranges:

$$\sum_{\substack{n<N_k \\ \gcd(n,N_k)=1}} n^{s-1} = \sum_{d|N_k} \mu(d) d^{s-1} \sum_{m<N_k/d} m^{s-1}$$

Euler-Maclaurin approximates the inner sum:

$$\sum_{m=1}^{M} m^{s-1} = \frac{M^s}{s} + \frac{M^{s-1}}{2} + O(M^{s-2})$$

yielding the Euler product times zeta, with error from truncation. This links the wave to partial zeta sums.

Corollary 2 (Partial L-Function)**.** *Define truncated L-function:*

$$L_k(s,\chi) = \sum_{\substack{n \le N_k \\ \gcd(n,N_k)=1}} \frac{\chi(n)}{n^s}$$

Then $L_k(s,\chi) = \mathcal{M}P_k^{(\chi)} \cdot (s+1)$*, providing explicit partial sums via the isolated wave.*

2.4 Explicit Computations for Small k

Example 1.7 (Complete Character Analysis for k=3)
For $N_3 = 30$, $\phi(30) = 8$ characters. Primitive ones listed with Gauss sums (e.g., mod 2: 1,-1 with $\sqrt{2}$). For m=1:

$$c_1^{(3)} = \frac{1}{30}\left[\chi_0(1)\tau(\chi_0) + \chi_1(1)\tau(\chi_1) + \cdots\right] = -\frac{1}{30}$$

This matches Ramanujan but adds character structure.

3 Implications and Future Directions

These proofs embed PWT in character theory, improving spectral bounds and linking to L-functions for twin prime insights (e.g., spectral correlations in Section 12.2.1 of the thesis suggest twin prime density via wave overlaps). As Tusk notes in V15.1, this opens doors to analytic extensions, potentially addressing conjectures like the Twin Prime Conjecture through "spectral analysis."

For visualizations, the thesis's Python code (Appendix C) generates wave plots—try running it for k=3 to see the periodic structure. PWT continues to evolve, bridging sieves and spectra in novel ways.