Unveiling the Complex Depths of Prime Wave Theory: Key Proofs from Version 15

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Abstract

Prime numbers have long fascinated mathematicians, appearing random yet governed by deep patterns. In the recently released "Prime Wave Theory: A Fourier-Analytic Perspective on the Sieve of Eratosthenes" (Version 15.1, authored by Tusk and dated October 1, 2025), a novel framework reimagines the ancient Sieve of Eratosthenes through wave interference. This thesis establishes the Prime Wave Theory (PWT) as a rigorous tool for prime identification, using periodic pulses to create a "prime wave" function $P_k(x)$ that interpolates the sieve's output.

The core intuition: the sieve eliminates multiples of primes by marking them as composite. PWT translates this into multiplicative wave pulses, where $P_k(n) = 1$ if n is coprime to the product of the first k primes ($N_k = \prod_{i=1}^{k} p_i$), and 0 otherwise. The continuous extension allows Fourier-analytic tools to probe deeper properties.

This article focuses on Part III of the research program excerpted from the thesis: "Complex Extension and Analytic Properties." We describe and present the key theorems and proofs, incorporating verified corrections for accuracy (e.g., growth bounds, zero characterizations). Visualizations help illustrate the wave-like nature of $P_k(x)$.

1 Analytic Continuation: Extending the Prime Wave to the Complex Plane

The foundation is Theorem 3.1, which extends the real-valued $P_k(x)$ to an entire function on $\mathbb{C}$.

Theorem 1 (Complex Prime Wave)**.** *The function*

$$P_k(z) = \prod_{i=1}^{k} \left[1 - \frac{1}{p_i} \sum_{j=0}^{p_i - 1} \cos\left(\frac{2\pi j z}{p_i} \right) \right]$$

is entire on $\mathbb{C}$, periodic with period N_k, and satisfies:

1. $P_k(\overline{z}) = \overline{P_k(z)}$ *(conjugate symmetry)*

2. $P_k(z + N_k) = P_k(z)$ *(periodicity)*

3. $|P_k(x + iy)| \leq \exp\left(2\pi|y| \sum_{i=1}^{k} \frac{p_i - 1}{p_i}\right) \leq \exp(2\pi k|y|)$ *(exponential growth)*

Proof Description: Each factor $\Psi_{p_i}(z) = 1 - \frac{1}{p_i} \sum_{j=0}^{p_i-1} \cos\left(\frac{2\pi j z}{p_i}\right)$ is entire, as cosine is entire. Symmetry and periodicity follow directly. The growth bound uses $|\cos(x + iy)| \leq \cosh(2\pi j|y|/p_i) \leq \exp(2\pi(p_i - 1)|y|/p_i)$ for the maximum j, leading to the product bound. This corrects earlier versions with π instead of 2π, providing a tighter estimate exponential in k.

This extension allows analytic tools like contour integration, revealing PWT's links to complex analysis.

2 Zeros in the Complex Plane: Where the Wave Vanishes

Theorem 2 (Complex Zeros)**.** *For $z = x + iy$ with $y \neq 0$, $P_k(z) = 0$ if and only if there exists i such that*

$$\sum_{j=0}^{p_i-1} e^{2\pi i j z/p_i} = p_i,$$

which occurs only when $z/p_i \in \mathbb{Z}$. All zeros are real integers with $\gcd(z, N_k) > 1$; no non-real zeros exist.

Proof Description: The sum is geometric:

$$\sum_{j=0}^{p_i-1} e^{2\pi i j z/p_i} = \frac{1 - e^{2\pi i z}}{1 - e^{2\pi i z/p_i}}.$$

It equals p_i only if $e^{2\pi i z/p_i} = 1$ (i.e., real integer multiple). For $y \neq 0$, the magnitude

$$\left|\frac{\sin(\pi z)}{\sin(\pi z/p_i)}\right| < p_i$$

strictly, due to $|\sin(\pi z)|^2 = \sin^2(\pi x) + \sinh^2(\pi y)$ and the growth of sinh. This disproves earlier lattice claims for complex zeros, confirming discreteness without accumulation.

Lemma 1. *For $y \neq 0$, $|P_k(x + iy)| > 0$.*

Corollary 1. *The zeros of $P_k(z)$ are exactly the integers n with $\gcd(n, N_k) > 1$.*

3 Rigorous Connection to Dirichlet L-Functions

Theorem 3 (L-Function via Mellin-Fourier)**.** *For primitive* $\chi \mod N_k$*,*

$$L_k(s,\chi) = \sum_{n=1}^{N_k} \frac{\chi(n)P_k(n)}{n^s} = \sum_{\substack{n \le N_k \\ \gcd(n,N_k)=1}} \frac{\chi(n)}{n^s}.$$

For $\Re(s) > 1$*, this equals the Euler product over primes beyond* p_k *plus* $O(N_k^{1-\Re(s)})$*.*

Proof Description: By multiplicativity and truncation, the error arises from high powers exceeding N_k. Corrected from the original (which swapped small/large primes), this uses character orthogonality for explicit computation via DFT in $O(N_k \log N_k)$ time.

Corollary 2. *The computation of* $L_k(s,\chi)$ *can be performed in* $O(N_k \log N_k)$ *operations using the Fast Fourier Transform.*

This bridges PWT to analytic number theory, enabling zeta-function approximations.

4 The Spectral Zeta Function: A Prime-Wave Variant

Definition 1. *The spectral zeta function is defined as*

$$\zeta_{P_k}(s) = \sum_{n=1}^{N_k} \frac{P_k(n)}{n^s},$$

the partial zeta over coprimes.

Theorem 4 (Functional Equation for ζ_{P_k})**.** *For the infinite analog,*

$$\zeta_{P_k}(s) = \zeta(s) \prod_{i=1}^{k} (1 - p_i^{-s}) \quad (\Re(s) > 1),$$

meromorphic with simple pole at $s = 1$*, residue* $\phi(N_k)/N_k$*.*

Proof Description: The Euler product for the incomplete zeta holds; the finite sum is entire, but the infinite has the pole. Residue calculation uses $\sum_{n \le X} n^{-s} \sim X^{1-s}/(s-1)$ scaled by density $\phi(N_k)/N_k$. Corrections fix finite/infinite confusion and residue errors.

5 Implications and Future Directions

These proofs demonstrate PWT's power: from sieve equivalence to complex analytics, offering tools for twin primes (via correlations) and beyond. As noted in the thesis, open questions include full Besov regularity (conjectured) and deeper twin prime links. PWT V15 paves the way for spectral sieving, potentially unlocking new prime insights.