Exploring Prime Wave Theory: A Fourier-Analytic View on the Sieve of Eratosthenes and Twin Primes

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Abstract

The quest to understand prime numbers has captivated mathematicians for centuries. From the ancient Sieve of Eratosthenes to modern conjectures like the Twin Prime Conjecture, primes remain a cornerstone of number theory. In a newly released thesis titled *Prime Wave Theory: A Fourier-Analytic Perspective on the Sieve of Eratosthenes* (Version 15.1, dated October 01, 2025), author Tusk introduces an innovative framework called Prime Wave Theory (PWT). This approach reimagines the sieve as a "Prime Wave"—a periodic binary signal constructed via multiplicative pulses, analyzed through Fourier methods.

PWT bridges discrete sieving with continuous analytic tools, offering fresh insights into prime distribution, regularity, and correlations. A key highlight is its application to twin primes (pairs like (3,5) or (11,13)), where spectral analysis reveals connections to the Hardy-Littlewood constant and gap statistics. Below, we delve into PWT's core ideas, focusing on the twin prime spectral analysis from the thesis's research program. We'll present and describe the corrected proofs, incorporating minor fixes for accuracy: adjusting a numerical constant in average gap estimates (from ∼1.7 to ∼9.6) and updating the twin prime count up to 2310 (from 52 to 69 pairs).

1 The Foundations of Prime Wave Theory

At its heart, PWT transforms the Sieve of Eratosthenes into a wave-like function. Let $p_1 = 2, p_2 = 3, \ldots, p_k$ be the first k primes, and $N_k = \prod_{i=1}^{k} p_i$ their product (the primorial). The discrete Prime Wave $P_k(n)$ is 1 if $\gcd(n, N_k) = 1$ (n coprime to all primes $\leq p_k$), and 0 otherwise—essentially marking potential primes after sieving up to p_k.

This is built recursively: Start with a pulse for each prime (periodic functions that are 1 except at multiples of p_i), then multiply them pointwise. The continuous extension $P_k(x)$ interpolates this over $[0, N_k)$, enabling Fourier analysis via Ramanujan sums.

The thesis's contributions include explicit Fourier representations, regularity in function spaces (Sobolev, Hölder, Besov), interpolation inequalities, and convergence theory. But the twin prime section stands out, using the correlation $C_2^{(k)}(x) = P_k(x)P_k(x+2)$ to probe pairs coprime to N_k—a sieve approximation to actual twin primes.

2 Twin Prime Spectral Analysis: Key Proofs and Descriptions

The twin prime part frames $C_2^{(k)}(n)$ as an indicator for "twin-coprime" pairs, analyzing its Fourier structure, mean density, and gaps. Below are the main theorems with corrected proofs, described for accessibility.

2.1 Exact Correlation Function and Fourier Expansion

Theorem 1 (Theorem 2.2'). *The Fourier coefficients of $C_2^{(k)}(x) = P_k(x)P_k(x+2)$ are*

$$\hat{C}_2^{(k)}(m) = \sum_{j=0}^{N_k-1} c_j^{(k)} \cdot c_{m-j}^{(k)} \cdot e^{-2\pi i j\cdot 2/N_k},$$

where $c_j^{(k)} = \frac{1}{N_k}\sum_{n=0}^{N_k-1} P_k(n)e^{-2\pi i jn/N_k}$ are the DFT coefficients of P_k.

Proof Description: Represent $P_k(x) = \sum_j c_j e^{2\pi i jx/N_k}$. Then,

$$C_2^{(k)}(x) = \left(\sum_j c_j e^{2\pi i jx/N_k}\right)\left(\sum_\ell c_\ell e^{2\pi i \ell(x+2)/N_k}\right) = \sum_j\sum_\ell c_j c_\ell e^{2\pi i(j+\ell)x/N_k} e^{2\pi i\ell\cdot 2/N_k}.$$

The m-th coefficient is

$$\hat{C}_2^{(k)}(m) = \frac{1}{N_k}\int_0^{N_k} C_2^{(k)}(x)e^{-2\pi i mx/N_k}dx = \sum_j\sum_\ell c_j c_\ell e^{2\pi i\ell\cdot 2/N_k}\cdot \delta_{j+\ell,m},$$

yielding the convolution with phase $e^{2\pi i(m-j)\cdot 2/N_k} = e^{2\pi i m\cdot 2/N_k}e^{-2\pi i j\cdot 2/N_k}$. The global phase doesn't affect the modulus. (Fix: Negative phase on j corrects the original sign error.) □

This spectral view allows analyzing correlations without direct enumeration, tying into Dirichlet characters for deeper arithmetic insights.

2.2 Emergence of the Twin Prime Constant

Theorem 2 (Theorem 2.3'). *The average $\mu_{C_2}^{(k)} = \frac{1}{N_k}\sum_{n=0}^{N_k-1} C_2^{(k)}(n)$ satisfies*

$$\mu_{C_2}^{(k)} = \frac{1}{2}\prod_{i=2}^{k}\frac{p_i-2}{p_i} \sim \frac{C_2 e^{-2\gamma}}{2\log^2 k},$$

where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and $C_2 \approx 0.66016$ is the Hardy-Littlewood twin prime constant.

Proof Description: By Chinese Remainder Theorem, the density is the product of local densities. For $p = 2$: Forbidden residue 0 mod 2, density 1/2 (n odd satisfies both). For $p_i \geq 3$: Forbidden 0 and -2 mod p_i, density $(p_i - 2)/p_i$.

Asymptotically:

$$\prod_{3 \leq p \leq p_k} \left(1 - \frac{2}{p}\right) = \prod_{3 \leq p \leq p_k} \left(1 - \frac{1}{p}\right)^2 \cdot \prod_{3 \leq p \leq p_k} \frac{1 - 2/p}{(1 - 1/p)^2} \sim \frac{e^{-2\gamma}}{\log^2 p_k} \cdot C_2,$$

since $\frac{1-2/p}{(1-1/p)^2} = 1 - \frac{1}{(p-1)^2}$. Including 1/2: the stated form. (Fix: Proper $p = 2$ handling and algebraic connection.) □

This shows how the sieve density approximates twin prime density ($\sim 2C_2/\log^2 x$), differing by a factor of 4 due to unsieved large primes.

2.3 Spectral Gap Analysis

Theorem 3 (Theorem 2.5'). *The average gap $\mathbb{E}[G_k] = 1/\mu_{C_2}^{(k)} \sim \frac{2\log^2 k}{C_2 e^{-2\gamma}} \approx 9.6 \log^2 k$.*

Proof Description: The density of points where $C_2^{(k)}(n) = 1$ implies average spacing as reciprocal, uniform by Chinese Remainder Theorem. □ (Fix: Corrected numerical constant using $e^{-2\gamma} \approx 0.315$, $C_2 \approx 0.66$, yielding ~ 9.6.)

Corollary 1 (Corollary 2.6'). *Variance $Var(G_k) \approx (\mathbb{E}[G_k])^2 \sim (9.6 \log^2 k)^2 \approx 92 \log^4 k$, assuming Poisson-like gaps.*

2.4 Higher-Order Correlations

Theorem 4 (Theorem 2.8'). *For constellation $\mathcal{H} = \{h_1, \ldots, h_r\}$,*

$$\mu_{\mathcal{H}}^{(k)} = \prod_{p_i \leq p_k} \left(1 - \frac{\nu_{p_i}(\mathcal{H})}{p_i}\right) \sim \mathfrak{S}(\mathcal{H}) \cdot \frac{e^{-r\gamma}}{\log^r p_k},$$

where ν_p is distinct residues mod p, and $\mathfrak{S}(\mathcal{H}) = \prod_p (1 - \nu_p/p)(1 - 1/p)^{-r}$. (Fix: Removed erroneous $r!$.)

2.5 Computational Example

Example 2.9': For $k = 5$ ($N_5 = 2310$), twin-coprime density ≈ 0.0584 (135 pairs). Actual twin primes ≤ 2310: 69 pairs. Efficiency $R_5 = 69/135 \approx 0.511$, close to heuristic ~ 0.508. (Fix: Updated count from 52 to 69; discrepancy for small k due to composites and asymptotics.)

3 Implications and Future Directions

PWT's spectral lens on twins provides a rigorous sieve framework, quantifying efficiency ($R_k \sim 4e^{2\gamma}/k^2$) and linking to deep conjectures. While finite k approximations need error terms, this work advances Fourier-analytic number theory. Future: Extend to polylogarithms or Riemann zeta connections.

This thesis (available via abstract) marks a promising step—primes as waves may unlock more secrets.