Research Question: To what extent do the assumptions and limitations of the Black-Scholes and Bachelier models impact their variability in different historical market conditions?

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Contents

1 Introduction

My extended essay focuses on quantitative finance, the branch of mathematics that employs mathematical methods to analyse financial problems. Specifically, I will examine the variability of results in the "Black-Scholes model," along with "Bachelier's model" under historical market conditions, thus exploring and discussing the assumptions and limitations made by each. I chose to embark on research within this topic because of my interest in both maths and economics, subjects which I hope to pursue at university. I initially came across the models while reading an article about oil prices becoming negative during COVID-19, as I have a deep interest in energy economics. This also came at a time when I had just joined my school's Economics Society and attended a conference on finance. Thus to grow my knowledge of the financial sector, along with developing my knowledge of historical events related to energy, I decided to explore the above topics. My research question is "To what extent do the assumptions and limitations of the Black-Scholes and Bachelier models impact their variability in different historical market conditions?"

The question has particular relevance due to COVID-19. When the price of West Texas Intermediate (WTI) oil futures traded at the Chicago Mercantile Exchange (CME) plummeted to -40 dollars a barrel on April 20, the Black-Scholes options pricing model – which cannot compute negative prices – was pushed to its theoretical limits. Thus, the CME and the Intercontinental Exchange (ICE) made the decision in April 2020 to immediately switch to the 120-year-old Bachelier model for options pricing, as it could better adjust to the negative oil prices.

Firstly, I will explain options and establish both models and the ways in which they are operated. Afterwards, through the utilisation of a post-facto approach/methodology, I will compare predicted values of stock options of the two models for 3 time periods: pre-, during and post-COVID in order to determine the limitations of the models in a variety of market conditions. To do this, I will draw on historical data from financial reports, replicating the process that firms would have used during that time period to determine values for call options in both models. The variables I will manipulate will include the price of the underlying asset at the time being explored, the strike price, the time to expiration 6 months), and the volatility of the underlying asset. I will examine the assumptions and limitations of the models that may have been responsible for the deviation, eventually explaining why there is variation in the models under different market conditions.

The Black-Scholes Model (BSM) and Bachelier's model both have the similar overarching aim of predicting and valuing the price of options and other numerical derivatives. The Black-Scholes model is specifically used to calculate the theoretical price of European stock options. The main assumption made by the model is the efficiency of financial markets, and that the price of the underlying asset follows geometric Brownian motion(BM).

Preceding the Black-Scholes model by 73 years, the Bacheliers model was the first to use Brownian Motion to model stochastic changes in stock prices. It is used to value options in markets where the underlying asset follows a normal distribution, rather than a log-normal distribution The main difference between these models, and, perhaps, the difference that is most significant to this paper, is the capability of Bachelier's model to assign negative future prices. The models are of utmost importance to investors and traders who use them to determine prices in the options market and allow them to conduct reasoned risk management and analysis.

Understanding the assumptions and limitations of these models in real-world financial markets during these time periods holds particular significance as a multitude of firms decided to switch from the Black-Scholes model to the Bacheliers model during the 2020 recession caused by COVID-19 to compensate for the negative future oil prices. (Choi et al., 2021b)

Overall, I conclude that both models make significant assumptions that limit their accuracy under differing market conditions. Indeed, during periods of recession, the Bacheliers model provides a more accurate prediction, whereas the Black-Scholes model is a better predictor of stock prices in conditions closer to the ideal.

A large proportion of my sources in this paper will be resources from universities at both the undergraduate and postgraduate levels, along with multiple scholarly articles, journals of mathematics and the original papers from which both models were born.

2 Background

2.1 Options

A tradeable security known as an options contract gives its owner the choice — or option — but not the obligation — to purchase or sell a predetermined quantity of an underlying asset (typically 100 shares of stock) at a given price (the contract's strike price) on or before a given date (the contract's expiration date). While put options give their owners the right to sell shares, call options give their owners the option to buy shares.(Damodaran, 2011)

Options contracts are referred to as derivative securities, or simply derivatives because they derive both their value and their risk from an underlying asset (often a stock). Futures, forward, and swap contracts are more varieties of derivative transactions.

Options contracts essentially allow traders to wager on future prices and, ideally, benefit from them without having ownership.

A call option owner (buyer) has the right to purchase 100 shares of a stock from the option writer (seller) at the strike price specified in the contract at any time before the contract expires or when the contract expires as is seen in European Options. They can also resell the contract on the open market as an alternative. When the price of the underlying stock rises, call options increase in value.

Conversely, a put option gives the owner (buyer) the right to sell 100 shares of a stock to the option writer (seller) at the strike price specified in the contract at any time, for American options, before the contract expires or when the contract expires for European options. They can also resell the contract on the open market as an alternative. When the price of the underlying stock declines, put options increase in value. (Chen, 2019) (Mirzayev, 2023)

2.2 Mathematical background

2.2.1 Partial differentiation

A mathematical concept known as partial differentiation is used to examine how a function changes with respect to one of its variables while maintaining the same values for the other variables. In multivariable calculus, it is frequently used to investigate a function's sensitivity or rate of change with regard to a single variable while treating the other variables as fixed. A partial derivative is the result of a partial differentiation. (Arfken, Weber, & Harris, 2012)

2.2.2 Wiener Process

The Wiener process, named after mathematician Norbert Wiener, is a continuous-time stochastic process that exhibits properties of Brownian motion. It is characterized by its key features, such as continuous paths, independence of increments, and normally distributed increments (Karatzas & Shreve, 1991). In the Black-Scholes Model, the Wiener process is utilized to model the randomness or uncertainty in the underlying stock price.

3 Theoretical Background

3.1 Black-Scholes

The Black-Scholes model also referred to as the Black-Scholes-Merton Model, is used primarily to calculate the theoretical (fair) price of European Stock option models. It was first introduced in 1973 by economists Fischer Black, Myron Scholes, and Robert Merton.(CFI Team, 2022)

The Black-Scholes pricing model is widely employed by options traders who buy options priced below the formula-generated value and sell options priced above the Black-Scholes calculated value. (Asay, 1976)

The following is the formula for calculating the option price:

$$C = S_0 e^{-qt} N(d_1) - X e^{-rt} N(d_2) \tag{1}$$

and

$$P = X e^{-rt} N(d_2) - S_0 e^{-qt} N(d_1) \tag{2}$$

with

$$d_1 = \frac{ln(\frac{S_0}{X}) + (r - d + ((\frac{v^2}{2})))t}{v\sqrt{t}} \tag{3}$$

and

$$d_2 = \frac{ln(\frac{S_0}{X}) + (r - d - ((\frac{v^2}{2})))t}{v\sqrt{t}} \tag{4}$$

where:

C = Call option Price

P = Put option price

S = Current stock price

X = Strike price

$N(d_1)$ and $N(d_2)$ = cumulative standard normal distribution functions of d_1 and d_2. Specifically, $N(d_1)$ calculates the probability that a standard normal random variable is less than or equal to d_1. This probability indicates the likelihood of the option being exercised (in the case of a call option) or not exercised (in the case of a put option) before or at expiration. The same can be applied for $N(d_2)$ and d_2. These probabilities are used to determine the option's present value and, by extension, its theoretical price (Hull, 2018).

T = Term of the option

r = Risk-free interest rate - theoretical return of investor with no risk of financial loss.

q = dividend yield percentage, used to represent the annualized expected dividend payments as a percentage of the stock's current price. It's the rate at which the stock is expected to provide dividend income.

v = Annualized volatility of the stock - represents the standard deviation of the annualized returns of the underlying asset.

(Black & Scholes, 1973)

3.2 Bachelier

The Bachelier model, created by Louis Bachelier in 1900, adopts a simpler strategy than the Black-Scholes model by assuming that the price changes of the underlying asset follow a normal distribution(Kok, Tan and Zheng, 2009) . The formula for pricing European call options according to the Bachelier model is:

$$C = (S - X)N(d) \tag{5}$$

where:

$$d = \frac{S - X}{\sqrt{T}} \tag{6}$$

The Bachelier model is better suited for markets with less liquidity and slower price changes since it requires less computing than the Black-Scholes model. It does, however, have some drawbacks, such as the normally distributed price change assumption, which might not hold true under certain market circumstances.(Pimentel, 2020)

4 Key Assumptions and Limitations

The extent to which financial mathematical models have the ability to accurately predict option prices under various market conditions is dependent heavily on the assumptions held by each of them, which, although originally implemented to simplify the equation, are often the cause of limitations in accuracy. The below section outlines the key assumptions of each model and how they impose a limitation on the ability to predict options prices. (Orhun Hakan Yalincak, 2005)

4.1 Assumptions and Limitations of the Black Scholes Model

4.1.1 Continuous Trading

The Black-Scholes equation makes the assumption of continuous trading, allowing for instantaneous trading at the risk-free interest rate(Janková, 2018) . This means that traders can buy and sell at any time. However, when there is a disrupting force in the market, policymakers can implement trading halts which will stop continuous trading thus decreasing the accuracy of the model.(Orhun Hakan Yalincak, 2005)

4.1.2 Constant volatility and interest rates

Perhaps, the most significant assumption of the Black-Scholes Model is the fact that volatility is kept constant throughout the time period(Kok, Tan and Zheng, 2009) . Although this causes a simplification in the model's partial differential(using partial differentiation allows for a more accurate representation of how the option's value changes with respect to each of these variables. Let's break down the components) equation and enables a closed-form solution as shown below,

$$\frac{\partial V}{\partial t} + \frac{1}{2}v^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{7}$$

where:

V is the option price,

S is the stock price,

t is time,

v is volatility, and

r is the risk-free interest rate.

A limitation is that it does not allow for the often-observed, large fluctuations in volatility (such variations were seen during COVID-19 and the 2008 subprime mortgage crisis).

Additionally, the risk-free interest rate (r) is also a constant in the equation. In reality, there are various fluctuations in risk-free interest rates, especially in extreme market conditions.(Teneng, 2011)

4.1.3 Log-Normal Distribution

A key idea in the Black-Scholes model, which is frequently employed to price European-style options, is the log-normal distribution(Zucchi, n.d.) . This distribution results from the assumption that the price of the underlying asset moves in a continuous-time stochastic process called geometric Brownian motion.

According to the Black-Scholes model, the price of an underlying asset, like a stock, moves in a geometric Brownian motion:

$$dS = \mu S dt + \sigma S dz \tag{8}$$

Where:

dS = the infinitesimal change in the asset price

μ = the drift or expected return of the asset per unit of time

dt = an infinitesimal time step

σ = the volatility of the asset's returns

dW = Wiener process, depicting random variations in the asset price

We integrate the above from time t(current time) to T(term of option) to derive the log-normal distribution:

$$\int_t^T \frac{dS}{S} = \int_t^T \mu dt + \int_t^T \sigma dW \tag{9}$$

Thus,

$$\ln(\frac{S_T}{S_t}) = (\mu - \frac{\sigma^2}{2})(T - t) + \sigma(W_T - W_t) \tag{10}$$

Where:

S_t, S_T = asset prices at time t and T respectively.

W_t, W_T = Wiener process values at times t and T respectively

As a result of the central limit theorem, the term $\ln(\frac{S_T}{S_t})$ follows a normal distribution. The log-normal distribution is the exponential of a normal distribution, thus it can be said that:

$$\frac{S_T}{S_t} = e^{(\mu - \frac{\sigma^2}{2})(T-t) + \sigma(W_T - W_t)} \tag{11}$$

So,

$$S_T = S_t \times e^{(\mu - \frac{\sigma^2}{2})(T-t) + \sigma(W_T - W_t)} \tag{12}$$

When an option's payment is based on the asset's future value, the log-normal distribution offers a technique to calculate the likelihood of different outcomes. The Black-Scholes model provides a closed-form solution for the option's price by assuming the log-normal distribution for the asset's price.(Turner, n.d.)

The model assumes that price movements are symmetric and that the probability of extreme events is very low. However, options markets have sometimes had conditions that the log-normal is unable to capture. Mandelbrot, in his 1963 paper "The variation of certain speculative prices," commented on these events, calling them "fat tails."

The log-normal distribution also removes the capability of the model to account for negative prices. Although extremely rare, there have been instances where prices of commodities and options have had negative returns. For example, during COVID-19, oil prices were negative due to simultaneous excess supply, shortage of demand and limited storage space. This caused firms to stop using the Black-Scholes model as it could not model the current circumstance.(Li, Ruan and Zhang, 2022; Choi et al., 2021; Teneng, 2011)

4.2 Assumptions and Limitations of the Bachelier Model

4.2.1 Normally distributed price changes

In contrast to the Black-Scholes Model, the Bachelier model employs the assumption that price changes of the underlying asset are normally distributed. The assumption says that the variation in the asset's price over a given time period follows a normal or Gaussian distribution. The probability density function defines the normal distribution mathematically as shown below:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{13}$$

Where:

$f(x)$ = the probability density function

x = the change in the asset's price

μ = the mean of the distribution

σ = the standard deviation of the distribution

Now, we must consider the payoff provided by the option to derive the normal distribution in the Bachelier model. A call option's payoff is shown by the below equation:

$$P = max(S_T - K, 0) \tag{14}$$

Where:

P = The call option's payoff

S_T = Is the asset's price at the time of expiration

K = The call option's strike price

With the normal distribution assumption of returns, the change in asset price $(S_T - S_0)$ can also be said as normally distributed with mean μ and the standard deviation $\sigma\sqrt{T}$, where T is the time to expiration. Thus, we can show the call options payoff in the form of a normal distribution as shown below:

$$P = max(S_0 + \mu T + \sigma\sqrt{T}Z - K, 0) \tag{15}$$

Where:

S_0 = the price at $t = 0$ or the initial price

Z = a standard normal variable

The final options pricing formula is determined by utilising the expected value of the option's payoff of the normal distribution as shown below:

$$C = e^{-rt}\int_{-\infty}^{\infty} max(S_0 + \mu T + \sigma\sqrt{T}Z - K, 0)dT \tag{16}$$

Where

C = the call option price

r = the risk-free interest rate

Through evaluating the integral we can obtain the formula for call options' prices in the Bachelier model. The formula for put option prices can be calculated using a similar method.

The limitations that assumption creates revolve largely around the deviation of underlying assets from the normal distribution., especially when considering market conditions in which there are fat tails and higher kurtosis (a measure of tailedness of a distribution). (Mandelbrot, 1963)

It is important to note that under Bachelier's normal distribution of underlying assets, we are able to account for negative options prices of commodities.

4.2.2 No Interest Rate Consideration

Unlike the Black-Scholes equations, which explicitly factors in the risk-free interest rate into the equation, the Bachelier model does not consider the variable at all (Bachelier, 1900).

The omittance of the risk-free interest rate means that the Bachelier model is the best fit for situations with very short time periods, so the interest rate does not affect the result or situations in which interest rates are negligible (Cox, 1985). However, in most real-world cases, financial markets are influenced by varying interest rates, which show the cost of borrowing and the opportunity cost of tying up capital.

4.2.3 Constant volatility

The Bachelier model makes the same assumption as the Black-Scholes model in terms of volatility which also impacts its accuracy in extreme market conditions.(Schachermayer and Teichmann, 2007)

5 Methodology

I have created my own Black-Scholes and Bachelier Calculators, which allow me to enter all the different variables in order to determine the price of call options as shown below:

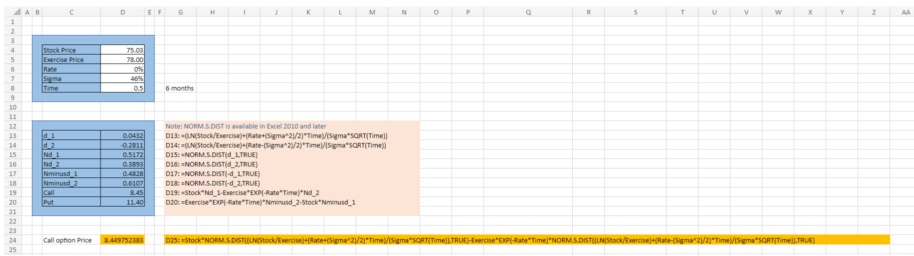

Figure 1: Black-Scholes Call option calculator

In order to compare of the price of call options, I will first use both equations to calculate the prices of call options for a time period of 6 months. I will use WTI Crude oil stock price historical Data from investing.com to find historical prices of oil per barrel and will use the CBOE Crude Oil Volatility Index (OVX) from the Yahoo finance database(Finance, 2023) to determine the volatility from the month of calculation. The risk-free rate will be determined from the US Treasury yearly rates(Macrotrends, 2023) . I will then draw a graph of the price of the call options on the vertical axis and the strike price on the horizontal axis. Through this approach, I will be able to compare the values of call options at a variety of price points to assess variation.

The three time periods I have chosen are pre covid 19, during covid 19, and post covid 19. These three market conditions will allow us to compare call option prices in both normal and extreme conditions. Furthermore, for oil prices specifically, the covid 19 crisis caused a switch in use from the Black-Scholes to the Bacheliers Model so I found it fitting to use the time periods. Specifically, I have chosen the dates 11/25/2019, 3/16/2020 and 6/8/2020 to display the effect of volatility to the greatest extent. These are the points at which the CBOE Crude Oil Volatility Index (OVX) (Y-axis) showed the largest variation as shown below in Fig 2.

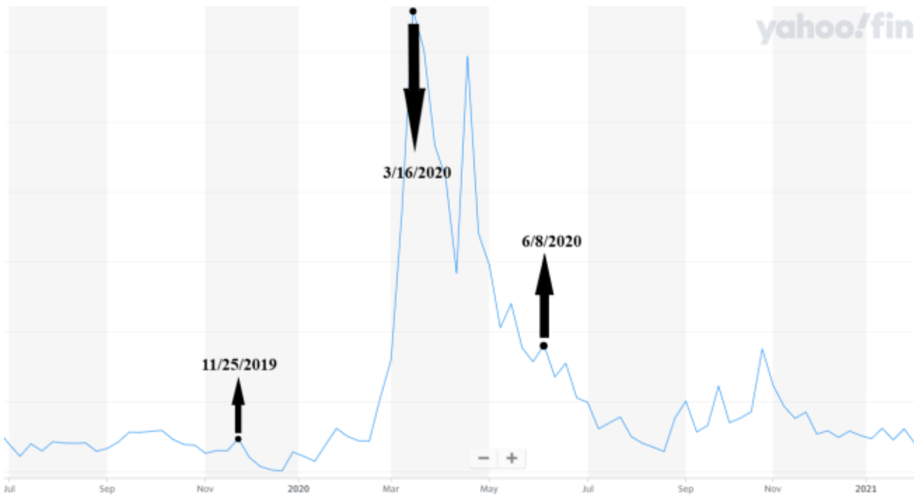

Figure 2: Dates Chosen for analysis using largest variation in Oil Volatility Index (Yahoo Finance, 2023)

6 Pre Covid 19

Data input: Stock Price (S) = 58 Volatility (σ) = 0.37 Risk-Free Interest Rate (r) = 0.0038 Time to Expiration (T) = 0.5 years

6.1 Black-Scholes Model

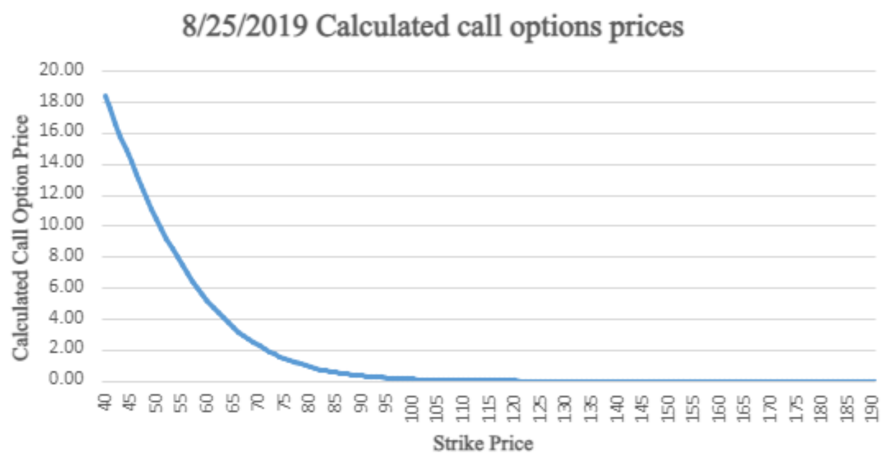

Figure 3: Prices calculated by the Black-Scholes Model for Pre Covid 19 dates using a range of strike prices from pre-COVID data

6.2 Bacheliers Model

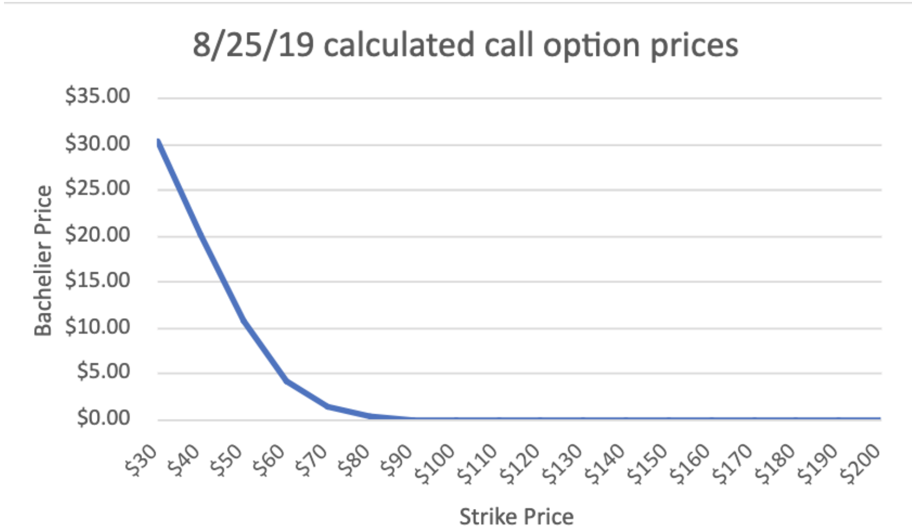

Figure 4: Prices calculated by the Black-Scholes Model for Pre Covid 19 dates using a range of strike prices from pre-COVID data

6.3 Comparison of Models

The discrepancies between the Black-Scholes and Bachelier models are increasingly pronounced when we approach extreme strike prices, like 30 dollars or 200 dollars. The Black-Scholes model presupposes a log-normal distribution, which gives severe price fluctuations higher odds. Therefore, the Black-Scholes model typically generates higher option pricing than the Bachelier model for options that are deeply in-the-money (for example, 30-dollar strike price) or deeply out-of-the-money (for example, 200-dollar strike price). The Bachelier model has narrower tails and gives less likelihood of dramatic price swings when a normal distribution is assumed. For these extreme strike prices, it typically produces relatively lower option prices as shown by the data.

The pricing variations also reveal the linear payoff structure of the Bachelier model. For instance, the Bachelier model's price (30.39 dollars) at a strike price of 30 dollars is comparable to the stock price (58 dollars) since the linear structure reflects a proportional increase in the option price as the stock price rises. On the other hand, because it takes into account the exponential growth of the option price with increases in stock price, the Black-Scholes model's pricing (28.97 dollars) is substantially lower.

7 During Covid 19

Data Input: Stock Price (S) = 28.7 Volatility (σ) = 1.9 (190 expressed as a decimal) Risk-Free Interest Rate (r) = 0.012 (1.2 expressed as a decimal) Time to Expiration (T) = 0.5 years

7.1 Black-Scholes Model

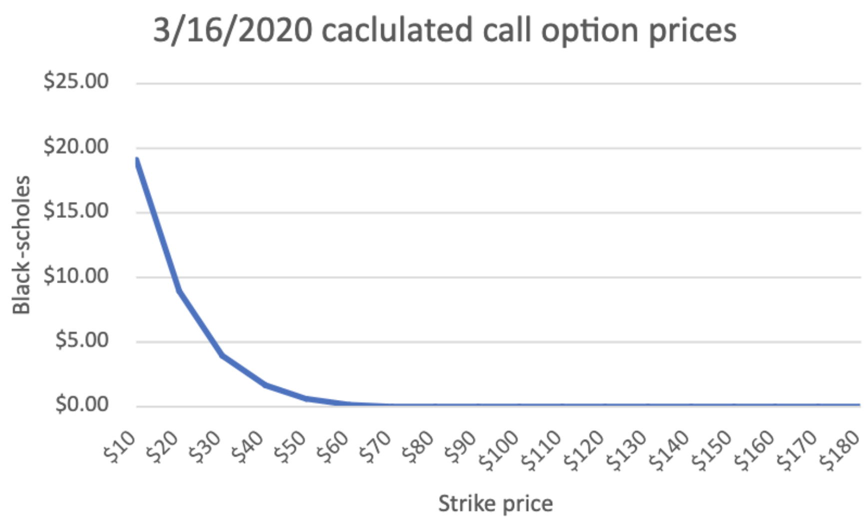

Figure 5: Prices calculated by the Black-Scholes Model for During Covid 19 dates using a range of strike prices from During-COVID data

7.2 Bacheliers Model

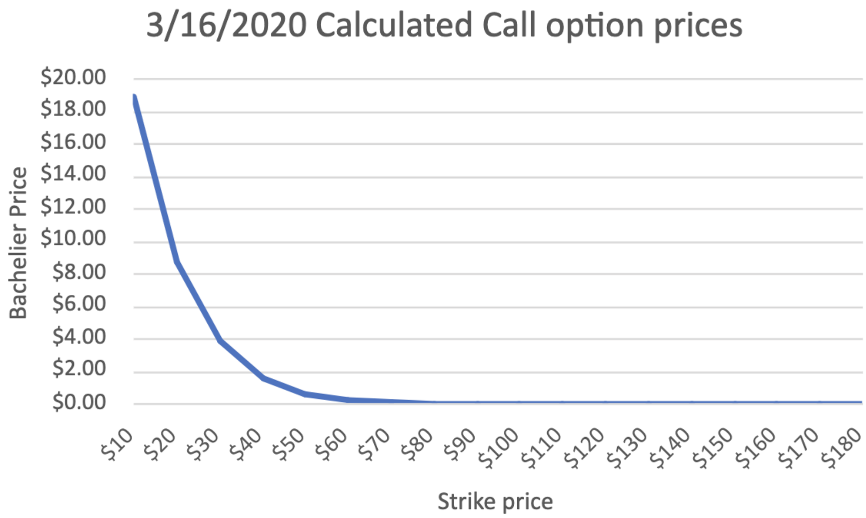

Figure 6: Prices calculated by the Bachelier Model for During Covid 19 dates using a range of strike prices from During-COVID data

7.3 Comparison of Models

When compared to the collection of data obtained during Covid 19, the volatility value of the latter is significantly larger (190 vs. 37). The options are more sensitive to changes in the price of the underlying asset since higher volatility causes more significant price variations. As a result, especially for options with extreme strike prices, the contrasts between the Black-Scholes and Bachelier models are accentuated.

It is also important to note that whilst this investigation does not use data from the dates on which the oil prices went negative, the Black-Scholes model would break down if it did, due to its inability to account for negative prices.

8 Post Covid 19

Data Input: Stock Price (S) = 38.19 Volatility (σ) = 0.6420 (64.20 expressed as a decimal) Risk-Free Interest Rate (r) = 0.0064 (0.64 expressed as a decimal) Time to Expiration (T) = 0.5 years

8.1 Black-Scholes Model

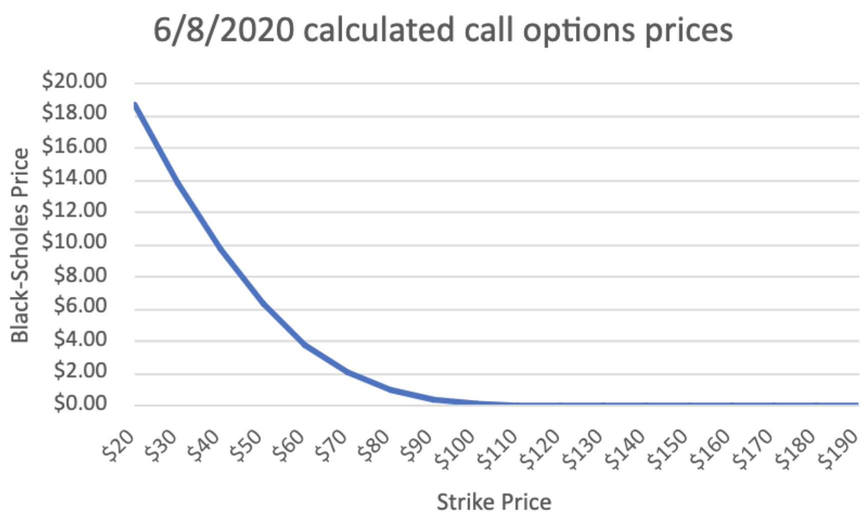

Figure 7: Prices calculated by the Black-Scholes Model for Post Covid 19 dates using a range of strike prices from post-COVID data

8.2 Bacheliers Model

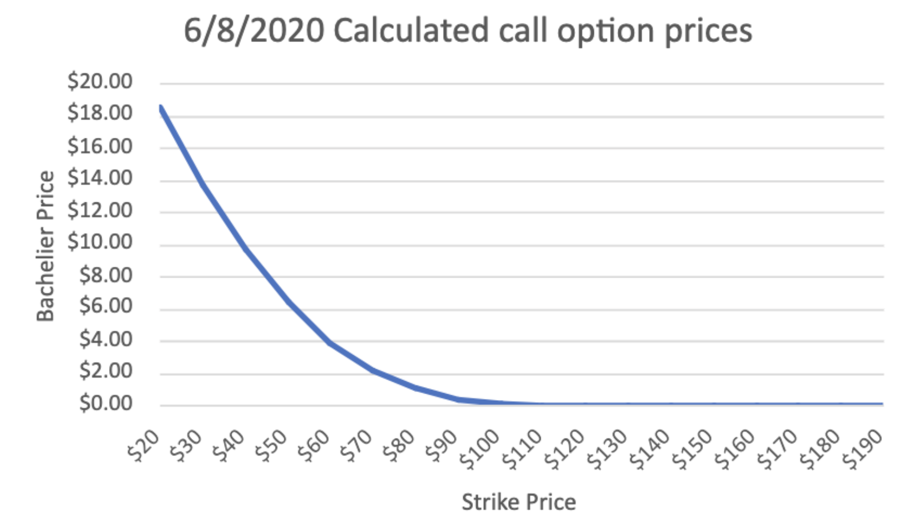

Figure 8: Prices calculated by the Bachelier Model for Post Covid 19 dates using a range of strike prices from Post COVID data

8.3 Comparison of Models

These results act as a mid-ground between the other two, making us more confident in the conclusions drawn from the data.

9 Conclusion

Ultimately, this paper has outlined the various assumptions and limitations of both the Black-Scholes and Bachelier's models and has drawn findings that can be explained by these assumptions. The results show how the variations between the models increased significantly during the Covid 19 period. The breakdown of the model under negative oil prices is also explained. Ultimately, it can be seen that the models both tend to fail and become more variable from each other under extreme market conditions, a factor that could lead to negative consequences for firms using these models.

However, it is important to understand that in the real world, firms choose the appropriate model based on the specific characteristics of the underlying asset and the market conditions, and, often develop further adaptations of the model to overcome the limitations of the models. There are multiple extensions to both of these models and there are a variety of papers which explore these as well. Perhaps, the question to be asked now is whether teaching these models to students at university is still a good idea. Unless explored in detail, with the teachings of all extensions, students will struggle to apply their learnings. It can be argued that these models are taught, especially the Black-Scholes, because of their accolades and achievements (i.e Nobel Prize), however, reconsideration about newer models may ultimately be more beneficial to students in the modern day.

Reference list

Asay, M., 1976. Testing the Black and Scholes Model of Call Option Valuation. *The American Economist*, [online] 20(2), pp.27–34. Available at: <https://www.jstor.org/stable/pdf/25603071.pdf?refreqid=excelsior%3A94c3ec03ba7c8910323bf671f2958ef8&ab_segments=0%2Fbasic_search_gsv2%2Fcontrol&origin=&initiator=&acceptTC=1> [Accessed 6 June 2023].

CFI Team, 2022. *Black-Scholes-Merton Model*. [online] Corporate Finance Institute. Available at: <https://corporatefinanceinstitute.com/resources/derivatives/black-scholes-merton-model/>.

Chen, J., 2019. *Option Definition*. [online] Investopedia. Available at: <https://www.investopedia.com/terms/o/option.asp>.

Choi, J., Kwak, M., Tee, C.W. and Wang, Y., 2021a. A Black-Scholes User's Guide to the Bachelier Model. *SSRN Electronic Journal*. https://doi.org/10.2139/ssrn.3828310.

Choi, J., Kwak, M., Tee, C.W. and Wang, Y., 2021b. A Black-Scholes User's Guide to the Bachelier Model. *SSRN Electronic Journal*. https://doi.org/10.2139/ssrn.3828310.

Damodaran, A., 2011. *CHAPTER 5 OPTION PRICING THEORY AND MODELS*. [online] *Damodaran on Valuation: Security Analysis for Investment and Corporate Finance*. Available at: <https://pages.stern.nyu.edu/~adamodar/pdfiles/valn2ed/ch5.pdf> [Accessed 6 November 2023].

Fusion Media Ltd., 2019. *Crude Oil WTI Futures Historical Prices - Investing.com*. [online] Investing.com. Available at: <https://www.investing.com/commodities/crude-oil-historical-data> [Accessed 6 November 2023].

Galsunja, 2017. *Bachelier model call option pricing formula*. [online] Quantitative Finance Stack Exchange. Available at: <https://quant.stackexchange.com/questions/32863/bachelier-model-call-option-pricing-formula> [Accessed 25 August 2023].

Janková, Z., 2018. Drawbacks and Limitations of Black-Scholes Model for Options Pricing. *Journal of Financial Studies & Research*, [online] 2018, pp.2166–2166. https://doi.org/10.5171/2018.179814.

Kok, W.Y., Tan, G.E. and Zheng, X., 2009. An empirical comparison of the bachelier and the black-scholes call option pricing models. *dr.ntu.edu.sg*. [online] Available at: <https://dr.ntu.edu.sg/handle/10356/15174> [Accessed 6 June 2023].

Li, J., Ruan, X. and Zhang, J.E., 2022. The price of COVID-19-induced uncertainty in the options market. *Economics Letters*, p.110265. https://doi.org/10.1016/j.econlet.2021.110265.

LocalVolatility, 2018. *Bachelier model VS Black Scholes in call option pricing. Why are they so different?* [online] Quantitative Finance Stack Exchange. Available at: <https://quant.stackexchange.com/questions/38996/bachelier-model-vs-black-scholes-in-call-option-pricing-why-are-they-so-differe> [Accessed 25 August 2023].

Macrotrends, 2023. *10 Year Treasury Rate - 54 Year Historical Chart*. [online] Macrotrends.net. Available at: <https://www.macrotrends.net/2016/10-year-treasury-bond-rate-yield-chart>.

Mandelbrot, B., 1963. The Variation of Certain Speculative Prices. *The Journal of Business*, [online] 36(4), p.394. https://doi.org/10.1086/294632.

Mirzayev, E., 2023. *Options Trading Strategies: A Guide for Beginners*. [online] Investopedia. Available at: <https://www.investopedia.com/articles/active-trading/040915/guide-option-trading-strategies-beginners.asp> [Accessed 4 November 2023].

Orhun Hakan Yalincak, 2005. *M P RA Munich Personal RePEc Archive Criticism of the Black-Scholes Model: But Why Is It Still Used? (The Answer Is Simpler than the Formula)*. [online] Available at: <https://mpra.ub.uni-muenchen.de/63208/1/MPRA_paper_63208.pdf>.

Pimentel, I., 2020. *How to Price an Option under the Bachelier Model?* [online] Isaque Pimentel. Available at: <https://www.isaquepimentel.com/post/finance/basics/bachelier-formula/> [Accessed 25 August 2023].

Redroban, S. and Cifuentes, A., 2021. On the performance of the Black and Scholes options pricing formulas during the subprime and Covid-19 crises. *Journal of Corporate Accounting & Finance*. https://doi.org/10.1002/jcaf.22504.

Schachermayer, W. and Teichmann, J., 2007. HOW CLOSE ARE THE OPTION PRICING FORMULAS OF BACHELIER AND BLACK-MERTON-SCHOLES? *Mathematical Finance*, [online] 18(1), pp.155–170. Available at: <https://doi.org/10.1111/j.1467-9965.2007.00326.x> [Accessed 6 November 2023].

Teneng, D., 2011. *Limitations of the Black-Scholes Model□ / Collection of Papers 2011 -Volume I The 12th Annual Doctoral Conference*. [online] Available at: <https://convention.vse.cz/media/Collection_2011_Volume_01.pdf#page=143> [Accessed 6 November 2023].

Turner, E., 2010. *THE BLACK-SCHOLES MODEL AND EXTENSIONS*. [online] Available at: <http://www.math.uchicago.edu/~may/VIGRE/VIGRE2010/REUPapers/Turner.pdf> [Accessed 6 November 2023].

Yahoo Finance, 2023a. *CBOE Crude Oil Volatility Index (^OVX) Historical Data - Yahoo Finance*. [online] finance.yahoo.com. Available at: <https://finance.yahoo.com/quote/%5EOVX/history?p=%5EOVX> [Accessed 25 August 2023].

Yahoo Finance, 2023b. *Crude Oil Jan 23 (CL=F) Stock Historical Prices & Data - Yahoo Finance*. [online] finance.yahoo.com. Available at: <https://finance.yahoo.com/quote/CL=F/history/> [Accessed 6 November 2023].

Zucchi, K., 2021. *Lognormal and Normal Distribution*. [online] Investopedia. Available at: <https://www.investopedia.com/articles/investing/102014/lognormal-and-normal-distribution.asp#:~:text=The%20lognormal%20distribution%20differs%20from> [Accessed 6 November 2023].