Differential Equation Board Question 2017

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(1) Solve the differential equation $(\tan^{-1} x - y)\,dx = (1 + x^2)\,dy$. 2017, 4 Marks

(2) Find the particular solution of the differential equation $(x - y)\frac{dy}{dx} = (x + 2y)$, given that $y = 0$ when $x = 1$. 2017, 6 Marks

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(1) Solve the differential equation $(\tan^{-1} x - y)\,dx = (1 + x^2)\,dy$. 2017, 4 Marks

Sol

Given $\frac{dy}{dx} = \frac{\tan^{-1}x - y}{1+x^2}$

Linear diff. eqⁿ

$\frac{dy}{dx} + Py = Q$

P & Q are constants or fn of x only.

Integrating factor $I.F = e^{\int P dx}$

Solution $\Rightarrow y(I.F) = \int Q.(I.F)\,dx + C$

$$\frac{dy}{dx} + \frac{y}{1+x^2} = \frac{\tan^{-1}x}{1+x^2}$$

On comparing

$P = \frac{1}{1+x^2}$, $Q = \frac{\tan^{-1}x}{1+x^2}$

$$I.F = e^{\int P dx}$$
$$= e^{\int \frac{1}{1+x^2}dx} = \boxed{e^{\tan^{-1}x}}$$

Solution

$$y\,(\underbrace{e^{\tan^{-1}x}}_{I.F}) = \int \left(\frac{\tan^{-1}x}{1+x^2}\right)(\underbrace{e^{\tan^{-1}x}}_{I.F})\,dx + C$$

Put $t = \tan^{-1}x$

$\frac{dt}{dx} = \frac{1}{1+x^2}$

ILATE

$$y\left(e^{\tan^{-1}x}\right) = \int \frac{t}{\cancel{(1+x^2)}} e^t \cancel{(1+x^2)}\,dt + C$$

$$= \int \underset{I}{t} \,.\, \underset{II}{e^t}\,dt + C$$

apply integration by parts.

$$= t.e^t - \int 1.e^t\,dt + C$$

$$y\left(e^{\tan^{-1}x}\right) = te^t - e^t + C$$

$$y\left(e^{\tan^{-1}x}\right) = (\tan^{-1}x - 1)\,e^{\tan^{-1}x} + C$$

4 Marks

$$\boxed{y = \tan^{-1}x - 1 + Ce^{-\tan^{-1}x}}$$ Ans.

(2) Find the particular solution of the differential equation $(x - y)\frac{dy}{dx} = (x + 2y)$, given that $y = 0$ when $x = 1$. 2017, 6 Marks

Sol $\frac{dy}{dx} = \frac{x+2y}{x-y}$

Homogeneous form of Diff eq^n. Put $y = vx$ & then it is converted into variable separable form.

Let $y = vx$ ✓

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

Putting this value in the eq^n

$$v + x\frac{dv}{dx} = \frac{x+2(vx)}{x-(vx)} = \frac{1+2v}{1-v}$$

$$x\frac{dv}{dx} = \frac{1+2v}{1-v} - v = \frac{1+2v-v+v^2}{1-v} = \frac{v^2+v+1}{1-v}$$

$$\frac{1-v}{v^2+v+1}\,dv = \frac{1}{x}\,dx$$

Integrating both sides

$$\int \frac{1-v}{v^2+v+1}\,dv = \int \frac{1}{x}\,dx$$

Now do integration (This integration can be done with many ways this is one of them)

$$= -\frac{1}{2}\int \frac{(2v+1)-3}{v^2+v+1}\,dv = \int \frac{1}{x}\,dx$$

$$\Rightarrow -\frac{1}{2}\int \frac{2v+1}{v^2+v+1}\,dv + \frac{3}{2}\int \frac{1}{v^2+v+1}\,dx = \int \frac{1}{x}\,dx$$

let $t = v^2+v+1$

$\frac{dt}{dv} = 2v+1$

$$\Rightarrow -\frac{1}{2}\int \frac{\cancel{(2v+1)}}{t}\frac{dt}{\cancel{(2v+1)}} + \frac{3}{2}\int \frac{1}{\left(v+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\,dx = \log|x| + C$$

$$\Rightarrow -\frac{1}{2}\log|v^2+v+1| + \frac{3}{2}\times\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{v+\frac{1}{2}}{\sqrt{3}/2}\right) = \log|x| + C$$

$$\Rightarrow -\frac{1}{2}\log\left|\frac{y^2}{x^2} + \frac{y}{x} + 1\right| + \sqrt{3}\tan^{-1}\left(\frac{2\frac{y}{x}+1}{\sqrt{3}}\right) = \log|x| + C$$

$\Rightarrow$ Now we have $y=0$ when $x=1$

Put this value in the solution

$$-\frac{1}{2}\log\left|\frac{0}{1^2}+\frac{0}{1}+1\right|+\sqrt{3}\tan^{-1}\left(\frac{2\times\frac{0}{1}+1}{\sqrt{3}}\right)=\log|1|+C$$

$$0+\sqrt{3}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=0+C$$

$$\Rightarrow \quad C=\sqrt{3}\tan^{-1}\left(1/\sqrt{3}\right)$$

Solution (Particular)

$$\Rightarrow -\frac{1}{2}\log\left|\frac{y^2}{x^2}+\frac{y}{x}+1\right|+\sqrt{3}\tan^{-1}\left(\frac{2\frac{y}{x}+1}{\sqrt{3}}\right)=\log|x|+\sqrt{3}\tan^{-1}\left(1/\sqrt{3}\right)$$

Ans.