Differentiation Board Questions 2017

2017, 4 Marks

(1) If $x^y + y^x = a^b$, then find $\frac{dy}{dx}$.

(2) If $e^y(x+1) = 1$, then show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$.

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(1) If $x^y + y^x = a^b$, then find $\frac{dy}{dx}$.

Sol: Given $x^y + y^x = a^b$

differentiating w.r.t. x,

$$\frac{d}{dx}(x^y) + \frac{d}{dx}(y^x) = \frac{d}{dx}(a^b) \rightarrow \text{constant}$$

$$\Rightarrow \frac{d}{dx}(x^y) + \frac{d}{dx}(y^x) = 0 \quad \text{——— (1)}$$

Let $u = x^y$

$\therefore \log_e u = y \log_e x$

$$\frac{1}{u}\frac{du}{dx} = y \cdot \frac{1}{x} + \log x \frac{dy}{dx} \Rightarrow \frac{du}{dx} = u\left(\frac{y}{x} + \log x \frac{dy}{dx}\right)$$

or $$\frac{d}{dx}(x^y) = (x^y)\left(\frac{y}{x} + \log x \frac{dy}{dx}\right) \quad \text{——— (2)}$$

Now let $V = y^x \quad \therefore \log V = x \log y$

differentiating w.r.t x

$$\frac{1}{V}\frac{dV}{dx} = x \times \frac{1}{y}\frac{dy}{dx} + \log y \cdot 1$$

$$\Rightarrow \frac{dV}{dx} = V\left(\frac{x}{y}\frac{dy}{dx} + \log y\right)$$

or $$\frac{d}{dx}(y^x) = y^x\left(\frac{x}{y}\frac{dy}{dx} + \log y\right) \quad \text{——— (3)}$$

Putting the values from (2) & (3) into (1)

we get $$x^y\left(\frac{y}{x} + \log x \frac{dy}{dx}\right) + y^x\left(\frac{x}{y}\frac{dy}{dx} + \log y\right) = 0$$

$$\Rightarrow \left(x^y \log x + x y^{x-1}\right)\frac{dy}{dx} = -\left(y^x \log y + y x^{y-1}\right)$$

$$\frac{dy}{dx} = -\frac{y^x \log y + y x^{y-1}}{x^y \log x + x y^{x-1}}$$

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$$\Rightarrow \frac{dy}{dx} = -\frac{y^x \log y + y\, x^{y-1}}{x^y \log x + x\, y^{x-1}}$$ Ans. #

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(2) If $e^y(x+1) = 1$, then show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$. 2017, 4 Marks

Sol Given $e^y(x+1) = 1$ ✓

differentiating w.r.t. x,

$$e^y \frac{d}{dx}(x+1) + (x+1)\frac{d}{dx}e^y = \frac{d}{dx}1$$

$$e^y \times 1 + (x+1)\frac{d}{dy}e^y \frac{dy}{dx} = 0$$

$$e^y + (x+1)e^y \frac{dy}{dx} = 0$$

$$e^y + \frac{dy}{dx} = 0 \quad \text{——— (1)} \quad \Rightarrow e^y = -\frac{dy}{dx}$$

Again differentiating w.r.t. x,

$$e^y \frac{dy}{dx} + \frac{d^2y}{dx^2} = 0$$

$$\left(-\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right) + \frac{d^2y}{dx^2} = 0$$

$$\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$$ Ans. #