Integration Questions in 2017 Board Exam (Total Marks 11 Marks)

① Find : **1 Mark**

$$\int \frac{\sin^2 x - \cos^2 x}{\sin x \cos x}\, dx$$

② Find : **2 Marks**

$$\int \frac{dx}{5 - 8x - x^2}$$

③ Find : **4 marks**

$$\int \frac{\cos\theta}{(4+\sin^2\theta)(5-4\cos^2\theta)}\, d\theta$$

④ Evaluate : **4 marks**

$$\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x}\, dx$$

OR

Evaluate :

$$\int_1^4 \{|x-1| + |x-2| + |x-4|\}\, dx$$

① $\int \frac{\sin^2 x - \cos^2 x}{\sin x . \cos x} dx = -\int \frac{\cos^2 x - \sin^2 x}{\sin x \cos x} = -2\int \frac{\cos 2x}{2\sin x \cos x} dx = -2\int \frac{\cos 2x}{\sin 2x} dx$

$= -2\int \cot 2x\, dx = \cancel{-2} \times \left(\frac{1}{\cancel{2}}\right) \log|\sin 2x| + C$

$= -\log|\sin 2x| + C = \log|\sin 2x|^{-1} + C$

$= \log \frac{1}{|\sin 2x|} + C = \log|\operatorname{cosec} 2x| + C$ Ans. 1 Mark

Formula
$\int \cot x\, dx = \log|\sin x| + C$

② $\int \frac{1}{5-8x-x^2} dx = -\int \frac{1}{x^2+8x-5} dx = -\int \frac{1}{(x+4)^2-(4)^2-5} dx$

$= -\int \frac{1}{(x+4)^2-21} dx = \int \frac{1}{(\sqrt{21})^2-(x+4)^2} dx$

Formula
$\int \frac{1}{a^2-x^2} dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$

$= \frac{1}{2\sqrt{21}} \log\left|\frac{\sqrt{21}+x+4}{\sqrt{21}-x-4}\right| + C$ Ans. (2 Marks)

③ $I = \int \frac{\cos\theta}{(4+\sin^2\theta)(5-4\cos^2\theta)} dx$ (4 Marks)

$= \int \frac{\cos\theta}{(4+\sin^2\theta)(5-4(1-\sin^2\theta))} dx = \int \frac{\cos\theta}{(4+\sin^2\theta)(1+4\sin^2\theta)} d\theta$

let $\sin\theta = t$
$\frac{dt}{d\theta} = \cos\theta$

dt

$$= \int \frac{}{(4+\sin^2\theta)\,(5-4(1-\sin^2\theta))}$$

$\sin\theta = t$

$\frac{dt}{d\theta} = \cos\theta$

$$= \int \frac{dt}{(4+t^2)(1+4t^2)}$$

$$\frac{1}{(4+t^2)(1+4t^2)} = \left[\frac{(1+4t^2) - 4(4+t^2)}{(4+t^2)(1+4t^2)}\right] \times \left(-\frac{1}{15}\right)$$

$$= -\frac{1}{15} \int \left[\frac{\cancel{(1+4t^2)}}{(4+t^2)\cancel{(1+4t^2)}} - \frac{4\cancel{(4+t^2)}}{\cancel{(4+t^2)}(1+4t^2)}\right] dt$$

Formula

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$$= -\frac{1}{15}\int \frac{1}{4+t^2}\,dt + \frac{4}{15}\int \frac{1}{1+4t^2}\,dt$$

$$= -\frac{1}{15} \times \frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) + \frac{\cancel{4}}{15 \times \cancel{4}} \int \frac{1}{(1/2)^2 + t^2}\,dt$$

$$= -\frac{1}{30}\tan^{-1}\frac{\sin\theta}{2} + \frac{1}{15} \times \frac{1}{1/2}\tan^{-1} t/_{1/2} + C$$

$$= \frac{2}{15}\tan^{-1} 2\sin\theta - \frac{1}{30}\tan^{-1}\frac{\sin\theta}{2} + C \quad \text{Ans.}$$

(4) $I = \int_0^{\pi} \frac{x\tan x}{\sec x + \tan x}\,dx$ — (1) NCERT

Using Property

$$I = \int_0^{\pi} \frac{(\pi-x)\tan(\pi-x)}{\sec(\pi-x) + \tan(\pi-x)}\,dx = \int_0^{\pi}\left[\frac{(\pi-x)\tan x}{(\sec x)+(\tan x)}\right]dx \text{ — (2)}$$

Property

$$\int_0^a f(x) = \int_0^a f(a-x)\,dx$$

add (1) + (2)

$$2I = \int_0^{\pi} \frac{\pi\tan x}{(\sec x + \tan x)}\,dx$$

$$= \pi\int_0^{\pi} \frac{\sin x/\cos x}{1/\cos x + \frac{\sin x}{\cos x}}\,dx = \pi\int_0^{\pi}\frac{\sin x}{1+\sin x}\,dx$$

Property

$$\int_0^{2a} f(x) = 2\int_0^a f(x)\,dx$$

if $f(2a-x) = f(x)$

Using Property

$$\cancel{2}I = 2\pi\int_0^{\pi/2}\frac{\sin x}{1+\sin x}\,dx = \cancel{2}\pi\int_0^{\pi/2}\frac{(1+\sin x)-1}{(1+\sin x)}\,dx$$

$$I = \pi\int_0^{\pi/2}\left(1 - \frac{1}{1+\sin x}\right)dx = \pi\left[\int_0^{\pi/2} 1.\,dx - \int_0^{\pi/2}\frac{1}{1+\sin x}\,dx\right]$$

$$= \pi\left[\pi/2 - \int_0^{\pi/2}\frac{1}{(1+\sin x)} \times \frac{(1-\sin x)}{(1-\sin x)}\,dx\right.$$

$$I = \frac{\pi^2}{2} - \pi\int_0^{\pi/2}\frac{1-\sin x}{\cos^2 x}\,dx = \frac{\pi^2}{2} - \pi\int_0^{\pi/2}(\sec^2 x - \sec x\tan x)\,dx$$

$$= \frac{\pi^2}{2} - \pi\left[\tan x - \sec x\right]_0^{\pi/2} = \frac{\pi^2}{2} - \pi\left[\frac{\sin x}{\cos x} - \frac{1}{\cos x}\right]_0^{\pi/2}$$

$$= \frac{\pi^2}{2} - \pi \left[\tan x - \sec x\right]_0^{\pi/2} = \frac{\pi^2}{2} - \pi\left[\frac{\sin x}{\cos x} - \frac{1}{\cos x}\right]_0^{\pi/2}$$

$$= \frac{\pi^2}{2} + \pi\left[\frac{1-\sin x}{\cos x}\right]_0^{\pi/2} = \frac{\pi^2}{2} + \pi\left[\frac{(1-\sin x)}{\cos x} \times \frac{(1+\sin x)}{(1+\sin x)}\right]$$

$$= \frac{\pi^2}{2} + \pi\left[\frac{\cos x}{1+\sin x}\right]_0^{\pi/2} = \frac{\pi^2}{2} + \pi\left[0 - 1\right] = \frac{\pi^2}{2} - \pi$$

$$= \frac{\pi}{2}(\pi - 2) \text{ Ans}$$

4) $$I = \int_1^4 \{|x-1| + |x-2| + |x-4|\}\,dx$$

$$|x-a| = \begin{cases} x-a, & x > a \\ -(x-a), & x < a \\ 0, & x = a \end{cases}$$

$$\begin{cases} +(x-1) - (x-2) - (x-4) = -x+5 = 5-x, & 1 \le x < 2 \\ +(x-1) + (x-2) - (x-4) = x+1, & 2 \le x \le 4 \end{cases}$$

Property

$$\# \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^a f(x)\,dx$$

$$a \le c \le b$$

$$I = \int_1^2 (5-x)\,dx + \int_2^4 (x+1)\,dx$$

$$= \left[5x - \frac{x^2}{2}\right]_1^2 + \left[\frac{x^2}{2} + x\right]_2^4$$

$$= \left[(10-2) - \left(5 - \frac{1}{2}\right)\right] + \left[(8+4) - (2+2)\right]$$

$$= 3\tfrac{1}{2} + 8 = 11\tfrac{1}{2} \text{ Ans}$$