Derivation of the Black Scholes Formula (With Commentary)

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The Black Scholes Model has been the industry standard in options valuation ever since it was first derived. Ever wonder where it came from? Let's see.

Let's start with

$f = (t, W_t)$

Where:

t = time

W = Brownian Motion at time index t where $t \leq T$ and $t > 0$.

Brownian Motion is a physics concept that relates to the random movement of microscopic particles. It has been applied to finance to describe the "random walk theory" first developed by Louis Bachelier (1870 to 1946) as part of his PhD dissertation "The Theory of Speculation." It was later also used by Black, Scholes, and Merton when developing the formula that we explore in this paper.

We must note that stocks do not, in fact, move in a random walk. There are lots of factors that influence the movement of prices of securities, most of which are nonrandom. As of yet, mathematicians have had a hard time modeling exactly the movement of securities. One would argue this could be because investors are partially influenced by psychology, IE: behavioral economics. Indeed, the price change of securities seems to be both random and predictable simultaneously.

For now, lets assume Brownian Motion applies.

Recall the rules of stochastic calculus and differential form:

$(dt^2) = 0$

$(dt)(dW_t) = 0$

$(dW_t^2) = dt$

Therefore, f from time t → T can be written as:

$$df = f_t dt + f_w dW_t + f_{wt}(dt)(dW_t) + \frac{1}{2} f_{tt}(dt^2) + \frac{1}{2} f_{ww}(dW_t^2)$$

Using the rules of stochastic calculus, simplify to:

$$df = f_t dt + f_w dW_t + \frac{1}{2} f_{ww}(dt)$$

This can be rewritten in integral form:

$$f(T, W_T) - f(t, W_t) = \int_t^T f_t \, dt + \int_t^T f_w \, dtW_t + \frac{1}{2}\int_t^T f_{ww}(dt)$$

This is the differential of a time-dependent function of the variable. Let's take it and apply it to wealth management by using Itô's Lemma (2) function:

$$Y(T, W_T) - Y(t, W_t) = \int_t^T b_t \, dW_t$$

Where:

b = a series of investments made from t → T, indexed by t

Y = wealth accumulated as a consequence of this series of investments.

Therefore:

$$E[Y(T, W_T) - Y(t, W_t)] = \int_t^T b_t \, dW_t$$

And because of the law of iterated expectations and the definition of an integral,

$$E[Y(T, W_T) - Y(t, W_t)] = \int_t^T b_t \, dW_t = 0$$

So why would anyone invest if the expected value is 0? We have not yet factored in the drift term. This also implies that all the capital appreciation will be garnered from the drift term.

So, let's look at the drift.

$$Y(T, W_T) - Y(t, W_t) = \int_t^T a_t \, dt + \int_t^T b_t dW_t$$

Now the expected value of the investment outcome becomes the drift term. We can easily express this in differential form:

$$dY_t = a_t dt + b_t dW_t$$

The first term ($a_t dt$) is the drift term, the second ($b_t dW_t$) is the diffusion coefficient, and the series (dY_t) is an Itô process.

The differential from time t → T can be written as a Taylor series expansion:

$$df = f_t dt + f_y dY_t + f_{yt}(dt)(dY_t) + \frac{1}{2} f_{tt}(dt^2) + \frac{1}{2} f_{yy}(dY_t^2)$$

Let's substitute it in:

$$df = f_t dt + f_y(a_t dt + b_t dW_t) + f_{yt}(dt)(a_t dt + b_t dW_t) + \frac{1}{2} f_{tt}(dt^2) + \frac{1}{2} f_{yy}(dY_t^2)$$

Simplify:

$$df = f_y(a_t dt + b_t dW_t) + \frac{1}{2} f_{yy}((a_t dt + b_t dW_t)^2)$$

Substitute in the squared wealth differential:

$$df = f_t dt + f_y(a_t dt + b_t dW_t) + \frac{1}{2} f_{yy}((a_t dt + b_t dW_t)^2) + 2\big((a_t)(b_t)(d_t)(dW_t)\big) + (b_t)^2(dW_t)^2)$$

Simplify:

$$df = f_t dt + f_y dt + f_y(a_t dt + b_t dW_t) + \frac{1}{2} f_{yy}(b_t^2 dt)$$

Substitute new terms for drift and volatility:

$$dY_t = a_t Y_t dt + \beta_t Y_t dW_t$$

Where:

$$a_t = a_t Y_t$$

$$b_t = \beta_t Y_t$$

Rewrite as:

$$\frac{dY_t}{Y_t} = a_t dt + \beta_t dW_t$$

Substitute new terms for drift and volatility again, for the sake of simplicity and common notation:

$$\frac{dS}{S} = \mu dt + \sigma dW_t$$

Where:

$$\mu = a_t$$

$$\sigma = \beta_t$$

$$S_t = Y_t$$

You may or may not recognize this as Geometric Brownian motion, a special Itô process that has time-varying drift and volatility terms. The Black Scholes model assumes the underlying asset follows this process when in a hedged portfolio.

Further assumptions:

a) Unrestricted short selling
b) No dividends, taxes, or arbitrage opportunities
c) Continuous trading

These assumptions are not true in actuality. As we know, there are indeed restrictions on short-selling. Many stocks pay dividends, and taxes are an inevitability. Arbitrage opportunities exist (or else Warren Buffet wouldn't have ever made any money.) Also, stocks are not traded continuously.

Why are we making these assumptions if we know they're not true? We are attempting to remove noise from our calculations. We can add this stuff back in later.

$$dH = (dS)(P) + (Q)((dD(S, T - t))$$

Where:

P = Units of underlying asset

S = Asset price

Q = Units of derivative asset

H = Value of hedged portfolio

The function to value the derivative resembles the wealth function from Itô's Lemma (2). The derivative price and wealth are both time-dependent functions of an Itô process. The derivative's price is a time-dependent function of Geometric Brownian motion. Therefore, we can solve using Itô's Lemma (1):

$$dD(S, T - t) = D_s dS + D_t dt + D_{st}(dS)(dT) + \frac{1}{2}D_{ss}(dS^2) + \frac{1}{2}D_{tt}(dt^2)$$

Since we assume it follows Brownian Motion:

$$dS = S(\mu dt + \sigma S dW_t)$$

And

$$(dS)^2 = (\mu S dt + \sigma S dW_t)^2 = (\mu S dt)^2 + 2\big(\mu S^2 \sigma (dt)(dW_t)\big) + (\sigma S dW_t)^2$$

Which simplifies to:

$$(dS)^2 = \sigma^2 S^2 dt$$

Make the substitutions:

$$dD(S, T-t) = D_s(\mu S dt + \sigma S dW_t) - D_{T-t}dt + \frac{1}{2}D_{ss}(\sigma^2 S^2 dt)$$

$$dH = (dS)(P) + (Q)(D_s dS - D_{T-t}dt + \frac{1}{2}D_{ss}\sigma^2 S^2 dt)$$

When:

$$D_t = -D_{T-t}$$

Written more simply:

$$\Delta Portfolio\ Hedged\ Value = (Units\ of\ Asset)(\Delta Asset\ Price) + (Units\ of\ Derivative)(\Delta Derivative\ Price)$$

Returning to the math, we can make a variable substitution and eliminate the randomness of Geometric Brownian motion by constructing (at least temporarily) a risk-free portfolio:

$$dH = (dS)(-QD_s) + Q(D_s \text{ds} - D_{T-t}dt + \frac{1}{2}D_{ss}\sigma^2 S^2 dt)$$

Simplify to:

$$dH = Q(-D_{T-t}dt + \frac{1}{2}D_{ss}\sigma^2 S^2 dt)$$

So at time t:

$$H = (-QD_s)(S) + Q(D)(S, T-t)$$

What does this mean for the return of the portfolio? Since we eliminated the randomness associated with Geometric Brownian motion, there is no risk. However, since we removed randomness and risk, we also removed market compensation for risk, μ.

Recall the Capital Asset Pricing Model (CAPM):

$$\mu = r_f + \beta(E[r_m]) - r_f)$$

To satisfy the no-arbitrage assumption, portfolio return must be equivalent to the risk-free rate:

$$\frac{dH}{H} = \frac{(Q)(-D_{T-t}dt + \frac{1}{2}D_{ss}\sigma^2 S^2 dt)}{(Q)(-D_s S + D(S, T-t))} = rdt$$

Simplify:

$$\frac{dH}{H} = \frac{(-D_{T-t}dt + \frac{1}{2}D_{ss}\sigma^2 S^2 dt)}{(-D_s S + D(S, T-t))} = rdt$$

$$\frac{dH}{H} = \frac{(-D_{T-t}dt + \frac{1}{2}D_{ss}\sigma^2 S^2)}{(-D_s S + D(S, T-t))} = r$$

$$D_{T-t} + \frac{1}{2}D_{ss}\sigma^2 S^2 = r(-D_s S + D(S, T-t))$$

$$0 = -rD(S, T-t) + rD_s S - D_{T-t} + \frac{1}{2}D_{ss}\sigma^2 S^2$$

The portfolio we have constructed is riskless. Thus, the differential equation follows the same law of motion that a derivative would satisfy and market participants don't require compensation for risk in this case. To find the value of the derivative today we must discount the expected value of the derivative at the expected rate of return of the underlying asset.

We are making the assumption that removing randomness makes the portfolio "risk free." However, that's not exactly true. There are in fact, many risks that can affect a portfolio lacking randomness, obviously.

For now, let's go with the assumption that this is a "riskless" portfolio:

$$D = S_t, T-t) = e^{\mu(T-t)}E_t[D(S_T, 0)]$$

And:

$$D = S_t, T-t) = e^{\mu(T-t)}\hat{\mathrm{E}}_t[D(S_T, 0)]$$

Where:

Ê = the risk-neutral expected value of the derivative. Since the portfolio is riskless, it can be discounted at the risk-free rate.

Because of the definition of expected value, we know:

$$\hat{E}_T[D(S_T, 0)] = \int_X D\,(S_T, 0) d\hat{G}(s)$$

Where:

X = Support

To integrate, we must make variable substitutions and transformations to arrive at standard normal distribution, of which we will take the natural log to make lognormal distribution. Why lognormal? Because securities move in percentages. Also because of the definition of a differential equation.

Lognormal distribution makes sense, but this is one of the biggest flaws with the Black Scholes model. It assumes that stocks follow a Gaussian curve, but that simply is not true. During "normal market conditions" this is a close approximation. However, during turbulent markets, there seems to be an opportunity for arbitrage due to the movement of stocks no longer being lognormal.

In addition, as we discovered during the 2020 pandemic, the probability of a futures contract having a negative price is, in fact, greater than zero. This creates a problem, as addressed in one of my previous papers, because the natural log of a negative price is an imaginary number. Securities must have real number valuations, so this creates an issue for any holder of an options contract on a futures contract with a negative price. The probability of this event repeating is also greater than zero, as a negative futures contract price is possible whenever the commodity must be stored. When storage runs out, the futures contract can turn negative.

Let's set that aside for a moment and return to the Black Scholes derivation:

Using Itô's Lemma (2) we can solve for the transformed differential:

$$dln(S_t) = \left(\mu - \frac{\sigma^2}{t}\right)dt + \sigma dW_t$$

Rewrite in integral form:

$$ln(S_T) - \ln(S_t) = \int_t^T (\mu - \frac{\sigma^2}{t})\,dt + \int_t^T \sigma^2 dW_t$$

Solve using the laws of logarithms:

$$ln\left(\frac{S_T}{S_t}\right) \sim N\left(\int_t^T (\mu - \frac{\sigma^2}{t}\right)dt, \int_t^T \sigma^2 dt)$$

$$ln\left(\frac{S_T}{S_t}\right) \sim N\left(\left(\mu - \frac{\sigma^2}{t}\right)(T-t\right), \sigma(T-t))$$

$$\frac{S_T}{S_t} \sim Logn(e^{\left(\mu - \frac{\sigma^2}{2}\right)(T-t) + \frac{\sigma^2}{2}(T-t)}, \ldots)$$

$$\frac{S_T}{S_t} \sim Logn(e^{\mu(T-t)}, \ldots)$$

Since we are under the conditions of risk-neutrality established by our portfolio to satisfy our no-arbitrage assumption, μ=r where r (the risk-free rate) is an appropriate U.S. Treasury bond.

$$e^{\mu(T-t)} = e^{r(T-t)}$$

The risk-free bond price today is a function of time with a payoff of $1 can be written as:

$$B(t,T) = e^{-r(T-t)}$$

$$r(t,T) = -\ln\,(B(t,T))$$

So, let's make the substitutions into our risk-neutral log return functions:

$$\left(\mu - \frac{\sigma^2}{2}\right)(T-t) = \left(r - \frac{\sigma^2}{2}\right)(T-t) = -[ln(B(t,T) + \frac{1}{2}\sigma^2(T-t)]$$

$$E\left[ln\left(\frac{S_T}{S_t}\right)\right] = -(ln(B(t,T) + \frac{1}{2}\sigma^2(T-t))$$

$$var\left[ln\left(\frac{S_T}{S_t}\right)\right] = \sigma^2(T-t)$$

Therefore:

$$ln\left(\frac{S_T}{S_t}\right) \sim N(-[ln\left(B(t,T) + \frac{1}{2}\sigma^2(T-t)\right], \sigma^2(T-t)$$

And:

$$\hat{G}(S) = \hat{H}(S_T \leq S) = \hat{H}(ln\left(\frac{S_T}{S_t}\right) \leq (ln\left(\frac{S_T}{S_t}\right))$$

Now we can standardize this and convert into a standard normal distribution:

$$\hat{H}(z \leq \frac{ln\left(\frac{S_T}{S_t}\right) + [ln\big(B(t,T)\big) + \frac{1}{2}\sigma^2(T-t)]}{\sigma\sqrt{(T-t)}}) = \phi[\frac{ln\left(\frac{S_T}{S_t}\right) + [ln\big(B(t,T)\big) + \frac{1}{2}\sigma^2(T-t)]}{\sigma\sqrt{(T-t)}}]$$

Where:

$$\phi = Standard\ Normal\ Distribution$$

Simplify, to get the risk-neutral cumulative distribution function in terms of the standard normal distribution:

$$\phi(\frac{ln\left(\frac{B(t,T)S}{S_t}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}})$$

We now have everything we need to derive the Black Scholes formula. Let's go straight into deriving the call price as a discounted risk-normal expected value of the payoff:

$$C(S_t, T-t) = B(t,T)\hat{E}_t[max(S_t - X, 0)]$$

The support for the region is given by the payoff of the call, between the strike price and infinity, so because of the definition of expected value:

$$C(S_t, T-t) = B(t,T)\int_X^\infty (S-X)d\,\hat{G}(S)$$

$$C(S_t, T-t) = B(t,T)\int_X^\infty Sd\,\hat{G}(S) - B(t,T)X\int_X^\infty d\hat{G}(S)$$

The second integral is easy to compute:

$$B(t,T)X\int_X^\infty d\hat{G}(S) = B(t,T)X[\hat{G}(\infty) - \hat{G}(X)]$$

The price of a security clearly must be less than infinity. We can substitute ϕ for the risk-neutral cumulative distribution function, so:

$$B(t,T)X\int_X^\infty d\hat{G}(S) = B(t,T)X[1-\phi\left(\frac{ln\left(\frac{B(t,T)X}{S_t}\right)+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)]$$

Because standard distribution is symmetrical, we can conclude:

$$B(t,T)X\int_X^\infty d\hat{G}(S) = B(t,T)X[\phi\left(\frac{ln\left(\frac{S_t}{B(t,T)X}\right)+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)]$$

Now let's solve the first integral:

$$B(t,T)\int_X^\infty Sd\,\hat{G}(S)$$

$$z = \frac{ln\left(\frac{B(t,T)X}{S_t}\right)+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}$$

$$S = \frac{S_t}{B(t,T)}(e^{z\sigma\sqrt{(T-t)}-\frac{1}{2}\sigma^2(T-t)})$$

Rewrite, making substitutions:

$$\phi(z) = \int_{-\infty}^{z}\frac{1}{\sqrt{2\pi}}e^{\frac{-w^2}{2}}dw, d\phi(z) = \frac{1}{\sqrt{2\pi}}e^{\frac{-z^2}{2}}dz$$

Where:

$$\phi(z) = \hat{G}(S)$$
$$d\phi(z) = d\hat{G}(S)$$

Solve the new limits of integration:

$$S = X \Rightarrow z = \frac{ln\left(\frac{B(t,T)X}{S_t}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}$$
$$S = \infty \Rightarrow z = \infty$$

Rewrite the integral in terms of z.

$$B(t,T)\int_{\frac{ln\left(\frac{B(t,T)X}{S_t}\right)+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}}^{\infty} \frac{S_t}{B(t,T)} e^{z\sigma\sqrt{(T-t)}-\frac{1}{2}\sigma^2(T-t)} d\phi(z)$$

The function of bond returns cancels, which leaves us with a slightly smaller mess:

$$S_t\int_{\frac{ln\left(\frac{B(t,T)X}{S_t}\right)+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}}^{\infty} e^{z\sigma\sqrt{(T-t)}-\frac{1}{2}\sigma^2(T-t)} \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}} dz$$

Regroup the mess and factor:

$$S_t\int_{\frac{ln\left(\frac{B(t,T)X}{S_t}\right)+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{z\sigma\sqrt{(T-t)}-\frac{1}{2}\sigma^2(T-t)-\frac{z^2}{2}dz}$$

$$S_t\int_{\frac{ln\left(\frac{B(t,T)X}{S_t}\right)+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{(-\frac{1}{2}\left(z^2-2z\sigma\sqrt{(T-t)}+\sigma^2(T-t)\right))} dz$$

$$S_t \int_{\frac{ln\left(\frac{B(t,T)X}{S_t}\right)+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{(-\frac{1}{2}\left(z-\sigma\sqrt{(T-t)})^2\right)} dz$$

Recognize that we have a perfect square (thank God!), and make one variable substitution:

$$y = z - \sigma\sqrt{(t-t)}, z = y + \sigma\sqrt{(T-t)}, dy = dz$$

Solve for our new limits of integration:

$$z = \frac{ln\left(\frac{B(t,T)X}{S_t}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}} \Rightarrow y = \frac{ln\left(\frac{B(t,T)X}{S_t}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}} - \sigma\sqrt{(T-t)}$$

So:

$$S_t \int_{\frac{ln\left(\frac{B(t,T)X}{S_t}\right)+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}} - \sigma\sqrt{(T-t)}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy$$

$$z = \infty \Rightarrow y = \infty$$

Therefore, because of the definition of standard normal distribution:

$$S_t(\phi(\infty) - \phi\left(\frac{ln\left(\frac{B(t,T)X}{S_t}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right) - \sigma\sqrt{(T-t)}))$$

Obviously, infinity in the standard normal deviation is just one.

$$S_t(1 - \phi\left(\frac{ln\left(\frac{B(t,T)X}{S_t}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right) - \sigma\sqrt{(T-t)}))$$

Once again, since standard normal distribution is symmetrical, so:

$$S_t(\,\phi\left(\frac{ln\left(\frac{S_t}{B(t,T)X}\right)+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right))$$

Substitute that in for the first integral and rewrite:

$$C(S_{t,}T-t)=S_t\phi\left(\frac{ln\left(\frac{S_t}{B(t,T)X}\right)+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)-B(t,T)X\phi\left(\frac{ln\left(\frac{S_t}{B(t,T)X}\right)-\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)$$

Simplify to a form we all recognize:

$$\boldsymbol{V_{Call}=Se^{-r(T-t)}N(d_1)-Ke^{-r(T-t)}N(d_2)}$$

Where:

K = $B(t,T)X$ = Strike price of the option

r = Risk free rate of return (%)

T – t = Time to expiration, in years

S = Price of the underlying security, whether a stock or a futures contract

b = dividend yield of the underlying (%)

$$d_1=\frac{ln\left(\frac{S_t}{K}\right)+b+\frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{(T-t)}}$$

$$d_2=\frac{ln\left(\frac{S_t}{K}\right)-b\frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{(T-t)}}=d_1-\sigma\sqrt{(T-t)}$$

Now that we have seen the math, I don't need to go through it again. I will just write down the simplified formula for the price of a put:

$$\boldsymbol{V_{Put}=-Se^{-r(T-t)}N(-d_1)-Ke^{-r(T-t)}N(d_2)}$$