Integrals with Colored Pencils

You should also print out my Trig review as well as my limits and derivatives review sheets, as you might need some of that information.

Integral: Also known as an anti-derivative. The area under a curve. An integral "undoes" a derivative in differential equations. For now, you'll be asked to just "do it."

$$\int_a^b f(x)dx = \lim_{\substack{n\to\infty \\ \Delta x\to 0}} = \text{Definition of an integral in limit terms}$$

Remember these, or you'll be the reason why your math teacher drinks at home :

Definite Integral: has defined terms of integration and doesn't need +C.
Returns an exact answer.

Indefinite Integral: doesn't have defined terms of integration and NEEDS +C.
Returns a family of curves as an answer.

COMMON INTEGRALS

Integral of a constant	$\int C\,dx = 0$
Integral times a constant	$\int C\,f(x)dx = C\int f(x)dx$
Integral raised to a power **(POWER RULE)**	$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$
Adding integrals	$\int f(x) + g(x)dx = \int f(x)dx + \int g(x)dx$
Difference of integrals	$\int f(x) - g(x)dx = \int f(x)dx - \int g(x)dx$

INTEGRALS OF TRIG and LOG FUNCTIONS

$\int cos\, dx = sin\, x + C$	$\int csc^2\, dx = -\, cot\, x + C$
$\int sin\, dx = \; -cos\, x + C$	$\int sec\, x\, tan\, x\, dx = sec\, x + C$
$\int sec^2\, dx = tan\, x + C$	$\int csc\, x\, cot\, x\, dx = -cos\, x + C$
$\int \ln dx = \ln x + C$	$\int \mathrm{e}\, dx = \mathrm{e}\, x + C$

INTEGRATION BY SUBSTITUTION: The opposite or inverse of the chain rule.

You don't want to expand it for the same reason you don't want to expand functions using the chain rule of derivatives.

Example:

$$\int x(x^2 - 1)^{99}$$

Step 1) Transform it from integrating over "x" to integrating over "u."

Let: $u = x^2 - 1$ $\quad\quad \therefore du = 2x$

dx both sides: $du = 2xdx$ $\quad\quad \therefore dx = \frac{1}{2}du$

Step 2) Substitute u into the equation. (Ask the coach to put u in!)

$$\int xu^{99}(du) \cdot \frac{1}{2x}du$$

x cancels out, which leaves you with:

$$\frac{\frac{1}{2}}{100}u^{100} + \boldsymbol{C} = \frac{1}{200}(u^{100}) + \boldsymbol{C}$$

Step 3) REMEMBER TO PUT IT BACK INTO TERMS OF x BECAUSE IN CALCULUS, u AREN'T THE ANSWER!

$$\frac{1}{200}(x^2 - 1)^{100} + \boldsymbol{C}$$

INTEGRATION BY PARTS: The inverse of product rule.

All integration by parts problems will have this form:

$$uv - \int vdu$$

Example:

$$\int xe^{2x}dx \qquad Well, poop.$$

$Substitution\ won't\ work.$

Step 1) Bring u back into things because u were super helpful last time.

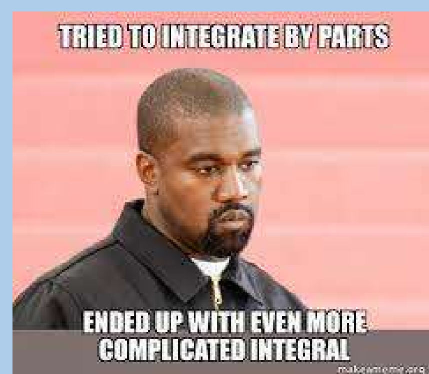
- **Your life coach will want to select the best u. Choose the u that has the smallest derivative but also makes sure that the integral of v is possible.**
- Otherwise, you'll end up with a mess, like Kanye over there.

Let: $u = x$, $\quad dv = The\ crap\ that's\ left\ over = e^{2x}dx$

$$\therefore \frac{du}{dx} = 1 \qquad \therefore du = dx$$

So:

$$\int dv = \int e^{2x}dx \qquad \therefore v = \frac{1}{2}e^{2x}$$



Step 2) Put it into the proper form for integration by parts.

$$The\ thing = uv - \int vdu$$

$$The\ thing = x\left(\frac{1}{2}e^{2x}\right) - \int \frac{1}{2}e^{2x}dx \quad = \frac{xe^{2x}}{2} - \frac{1}{2}\int e^{2x}dx \quad = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C$$

INTEGRATION BY TRIG SUBSTITUTION: Used to help with trig functions but can also be used to keep your equation in the real plane instead of letting it drift onto the complex plane.

The square root piece of junk looks a lot like… the Pythagorean Theorem. Make them into the hypotenuse or leg of a right triangle and magic!

If you have:	Substitute in:	Where:
$\sqrt{a^2 - x^2}$ $\quad (-\ sign\ means\ this\ is\ a\ leg)$	$x = a\sin\theta$	$1 - sin^2 = cos^2$
$\sqrt{a^2 + x^2}$ $\quad (+\ means\ this\ is\ a\ hypotenuse)$	$x = a\tan\theta$	$1 + tan^2 = sec^2$
$\sqrt{x^2 - a^2}$ $\quad (-sign\ means\ this\ is\ a\ leg)$	$x = a\sec\theta$	$sec^2 - 1 = tan^2$

TRIGONOMETRIC INTEGRATION

Two Cases:

$$\int \sin^m x \cos^n x \, dx \qquad \begin{cases} \textbf{Case 1}: \text{m } {}^{\text{and}}\!/_{\text{or}} \text{ n is ODD} \\ \textbf{Case 2}: \text{m and n are EVEN} \end{cases}$$

Case 1:

- If the power of sin is odd, strip off one factor of sin and use the Pythagorean identity to change $\sin^2 x = 1 - \cos^2 x$
- If the power of cos is odd, strip off one factor of cos and use Pythagorean identity to change
- If both are odd, choose the one that will give you a power of 2 if you can.

Case 2: Use half-angle formulas:

- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

Hot Tip:

I used to actually bring colored pencils to math tests so I could draw circles around parts as below, to make sure I didn't forget anything.

Example:

$$\int \cos^3(2x)\sin^5(2x)\,dx \quad = \quad \int \boxed{\cos^2(2x)}\sin^5(2x)\boxed{\cos(2x)\,dx}$$

$$= \int \boxed{1 - \sin^2(2x)}\sin^5(2x)\cos(2x)\,dx$$

$$Let\ u = 1 - \sin(2x)\ , \quad du = 2\cos(2x)\,dx\ , \quad \frac{du}{2} = \boxed{\cos(2x)\,dx}$$

$$= \int (1 - u^2)u^5\frac{du}{2} \quad = \quad \frac{1}{2}\int u^5 - u^7 du \quad = \frac{1}{2}\left[\frac{u^6}{6} - \frac{u^8}{8}\right]$$

$$= \frac{\sin^6(2x)}{12} - \frac{\sin^8(2x)}{16} + \boldsymbol{C}$$

INTEGRATION BY PARTIAL FRACTIONS: Commonly used when substitution no worky.

Recall from algebra : Every factorable polynomial fits a form :

ax + b or (ax^2+bx+c)

Step 1) Examine the degree of the equation.

Example:

$$\int \frac{4x^2 - 4x + 6}{x^3 - x^2 - 6x}dx \qquad \begin{cases} If\ numerator\ degree < denominator\ degree, use\ partial\ fractions) \\ If\ numerator\ degree\ \geq denominator\ degree, use\ long\ division\ first) \end{cases}$$

Step 2) Factor numerator and denominator into expressions of one of the following forms:

- Constants
- $(x-a)^r$ $where\ r\ is\ an\ integer\ \ (r \in \mathbb{Z})$
- $(x^2+bx+c)^q$ $where\ q\ is\ an\ integer\ \ (q \in \mathbb{Z})$

Simplify, or cancel common factors in numerator and denominator.

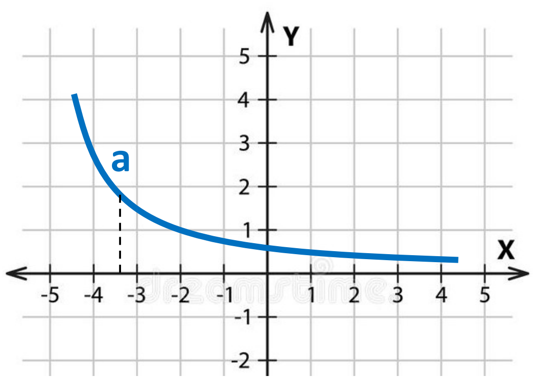
Step 3) Write in the following form:

$$\frac{A_1}{(x-a)}+\frac{A_2}{(x-a)^2}+\cdots+\frac{A_n}{(x-a)^n} \qquad and:$$

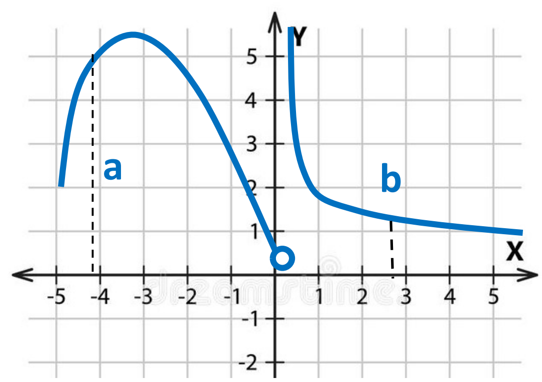
$$\frac{B_1x+C_1}{x^2+bx+c}+\frac{B_2x+C_2}{(x^2+bx+c)^{\ 2}}+\cdots+\frac{B_nx+C_n}{(x^2+bx+c)^n}$$

Step 4) Solve for A's B's and C's and integrate each expression separately.

IMPROPER INTEGRALS: There are two main types of improper integrals:



$$\int_a^\infty \frac{\mathbf{1}}{\boldsymbol{x^2}}\boldsymbol{dx}$$



$$\int_c^9 \frac{\mathbf{1}}{\sqrt{\boldsymbol{x}}}\boldsymbol{dx}$$

Interval is infinite

$$Let\ \infty = b, such\ that\ b > a$$

$$\therefore \int_a^b \frac{1}{x^2}dx \ \Rightarrow\ -\frac{1}{b}+1\ where\ b \to \infty$$

$$= \lim_{b\to\infty} -\frac{1}{b}+1 = 0+1 = 1$$

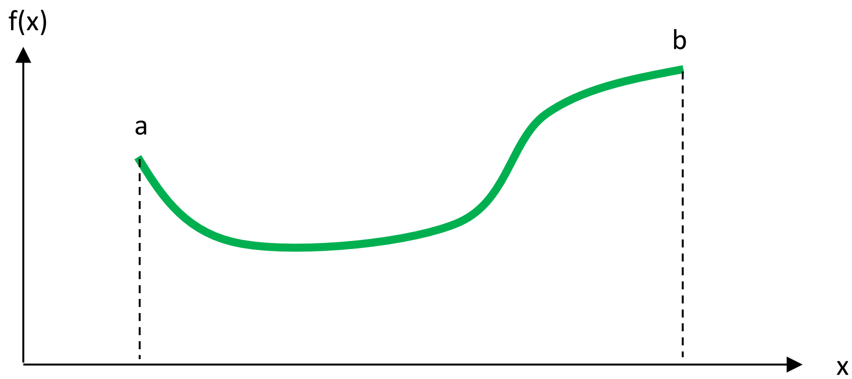
But, if f(x) is not bounded

We want $b \to 0^+$

$$\lim_{b\to 0^+}\int_b^9 \frac{1}{\sqrt{x}}dx \ = \ \lim_{b\to 0^+}\int_b^9 x^{-\frac{1}{2}}dx$$

Later, you'll learn about the Lebesgue integral, which is better for this function.

ARC LENGTH: The length of a curve segment



$$Arc\ Length = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

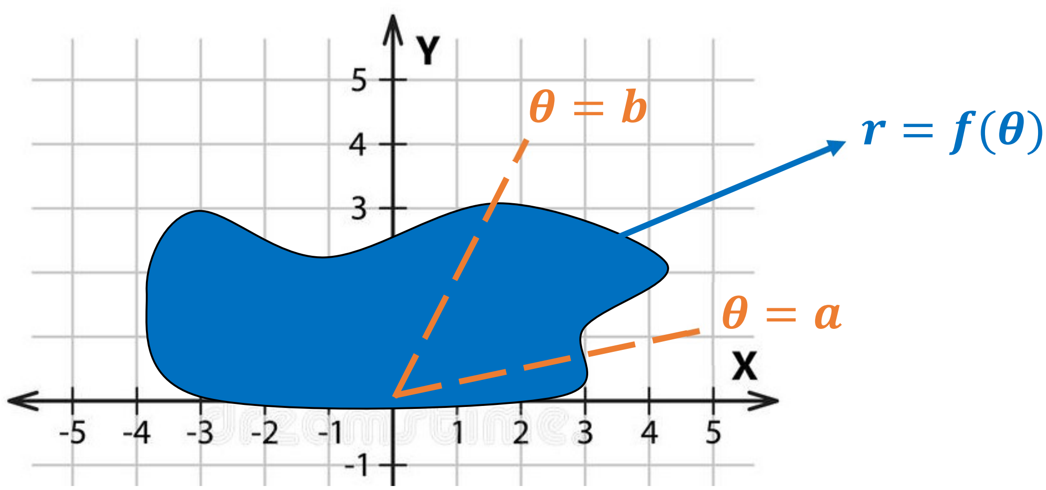
$$Surface\ area = (Arc\ length)(Circumference) = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx$$

POLAR VERSUS CARTESIAN COORDINATES

To convert from polar to cartesian: $(r, \theta) \rightarrow (x, y)$ $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

To convert from cartesian to polar: $(x, y) \rightarrow (r, \theta)$ $\begin{cases} r^2 = x^2 + y^2 & \Rightarrow HYPOTENUSE \\ tan\theta = \frac{y}{x} \end{cases}$

INTEGRATION IN POLAR COORDINATES: Find the area of the shaded region that lies between the orange dotted lines.



$$A = \int_a^b \frac{1}{2} r^2 d\theta \quad = \quad \int_a^b \frac{1}{2}[f(\theta)]^2 d\theta$$