Do Iterated and Lebesgue Integration with Highlighters!

Lots of people do Calculus 3 without understanding what iterated integration actually means. Don't do that! Math without application is super boring.

$$Volume = \iint_R f(x,y)dA$$

$$\Delta A = \Delta x \Delta y \equiv \Delta A = \Delta y \Delta x$$

$$dA = dxdy \equiv dA = dydx$$

If you're like me, you might appreciate an explanation of **what in tarnation** we're calculating with these functions.

The thing	What the thing could mean
$\int f(x)\,dx$	1) Area under $f(x)$ (two dimensions) 2) Mass of a one-dimensional wire
$\iint f(x,y)\,dA$	1) Volume under/inside $f(x,y)$ (three dimensions) 2) Mass of a two-dimensional plate iff the function is a mass density function 3) Charge densities
$\iiint f(x,y,z)\,dV$	1) Some measurement of a region under region $f(x,y,z)$ (four dimensions) 2) Mass of a three-dimensional surface iff the function is a mass density function
$\int_{\mathcal{L}} f(x)\,dx \begin{cases} Some\ crap \\ More\ crap \\ Other\ crap \end{cases}$	1) Area under a discontinuous or semicontinuous function 2) Only the area of a specific section of a function (like, only the section with positive values)

Honestly, this section will not be about integration as much as it is about showing some steps to solve these problems, the setup being the hardest and most important part. Keep in mind that some of them are impossible to solve.

I encourage students to bring colored pencils or highlighters when doing these problems, so that all the x and y terms match up. This will save you headaches later!

This is an example, provided you're good at coloring:

For rectangular regions $R: \begin{cases} (x, y) \mid a \leq x \leq b \\ \mid c \leq y \leq d \end{cases}$ you could write this integration two ways:

$$\int_{c}^{d} \int_{a}^{b} f(x, y)\, dx\, dy \qquad \qquad \int_{a}^{b} \int_{c}^{d} f(y, x)\, dy\, dx$$

You may notice this is a matching game.

Start with your first f and solve from the inside out, bottom up. Then match each f with the d. 🍆

These two expressions are equivalent. If you're going to solve the equation, choose the integral that is the easiest to solve.

Always solve the inside term first, so that variable GOES AWAY. Then and only then can you solve the second one.

$$Given: \iint_R x + y^2\, dA \;, \quad R: \{(x, y) | 0 \leq x \leq 1, \quad -1 \leq y \leq 2\}$$

$$\int_{-1}^{2} \int_{0}^{1} x + y^2\, dx\, dy \;\Rightarrow\; \int_{-1}^{2} \left[\frac{1}{2}x^2 + xy^2\right]_{x=o}^{x=1} dy \;\Rightarrow\; \int_{-1}^{2} \left(\frac{1}{2} + y^2\right) dy$$

$$\Rightarrow \left[\frac{1}{2}y + \frac{1}{3}y^3\right]_{y=-1}^{y=2} \;\Rightarrow\; \frac{9}{2}$$

Vertically Simple Region (x-simple): Has constant terms on the x-axis, so start with the y-axis.

$$Given: \iint_R f(xy)\, dA, \quad Vertically\ Simple\ iff: \int_{a}^{b} \int_{y=g_1(x)}^{y=g_2(x)} f(x, y)\, dy\, dx$$

Horizontally Simple Region (y-simple): Has constant terms on the y-axis, so start with the x-axis.

$$Given: \iint_R f(xy)\, dA, \quad Horizontally\ Simple\ iff: \int_c^d \int_{y=h_1(x)}^{y=h_2(x)} f(x,y)\, dx\, dy$$

Double Integrals in Polar Coordinates

Always start from the origin and go in the radius direction first. Or you can think about it as, always do your function part first and the constant part last. Since polar functions are in terms of theta, always do the function part first.

$$Given: \int_{\theta=\alpha}^{\theta=\beta} \int_{r=a}^{r=b} f(rcos\theta, rsin\theta) \quad \Rightarrow \quad dA = r dr d\theta$$

WTF? Where did the extra r come from??! We are taking the area of a pie wedge (dr), which is the arc length. Arc length = $r\theta$.

$$Volume = \iiint_T (x,y,z)\, dV$$

$$\Delta V = \Delta x \Delta y \Delta z \equiv \Delta V = \Delta y \Delta x \Delta z \equiv \cdots (6\ permutations), can\ be\ in\ any\ order$$

$$dV = dxdydz \equiv dV = dydxdz \equiv \cdots (6\ permutations), can\ be\ in\ any\ order$$

Z-Simple: Define the region between two "$z = f(x,y)$" functions. It's much easier to deal with a function on the (x,y) plane.

Y-Simple: Define the region between two "$y = f(x,z)$" functions.

X-Simple: Define the region between two "$x = f(y,z)$" functions.

Example:

$$f(x,y,z) = x \quad and \quad T: The\ region\ bounded\ by\ x = 0, y = 0, z = 0\ and\ x + y + z = 1$$

Try Z-simple first: $0 \le z \le 1 - x - y$. $\Rightarrow$ Region R is on the (x,y) plane.

Choose a direction: $0 \le y \le 1 - x$

Now the last one (must be between constants): $0 \le x \le 1$

Write out and highlight the integration:

$$\int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} x \, dz \, dy \, dx$$

You should get an answer of: $\frac{1}{24}$

FOR A CYLINDER-ISH OBJECT

(Doesn't have to be a perfect cylinder)

By integration, you'll be defining a mass of an object that has a height and two circular(ish) tops.

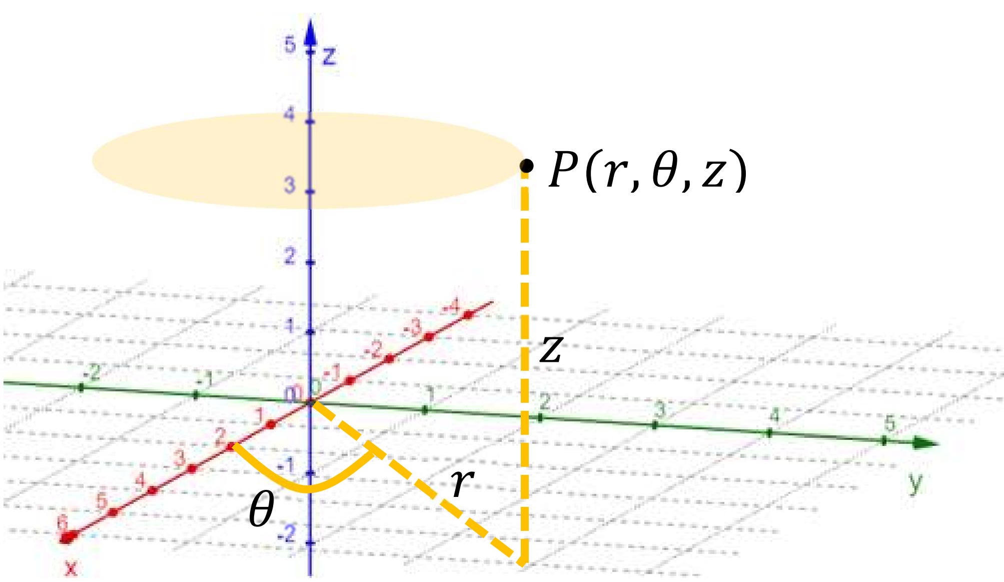
Well, poop. You're going to have to change everything into polar coordinates.

To convert from polar to cartesian: $(r, \theta) \rightarrow (x, y) \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

Secret: Everything that is Z-simple with polar (x,y) is a cylinder.

With cylinders, the thing must be done in a specific order. You can't change it around.

$$\int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{F_1(r\cos\theta, r\sin\theta)=}^{F_2(r\cos\theta, r\sin\theta)=} f(r\cos\theta, r\sin\theta, z) \; r \; dz \, dr \, d\theta$$



Often, we will be looking for specific points, $(0,0,0)$, which is on the bottom of the cylinder, along the (x,y) plane. We will also care about "P", which is defined as (r, θ, z) and will give the height of the cylinder.

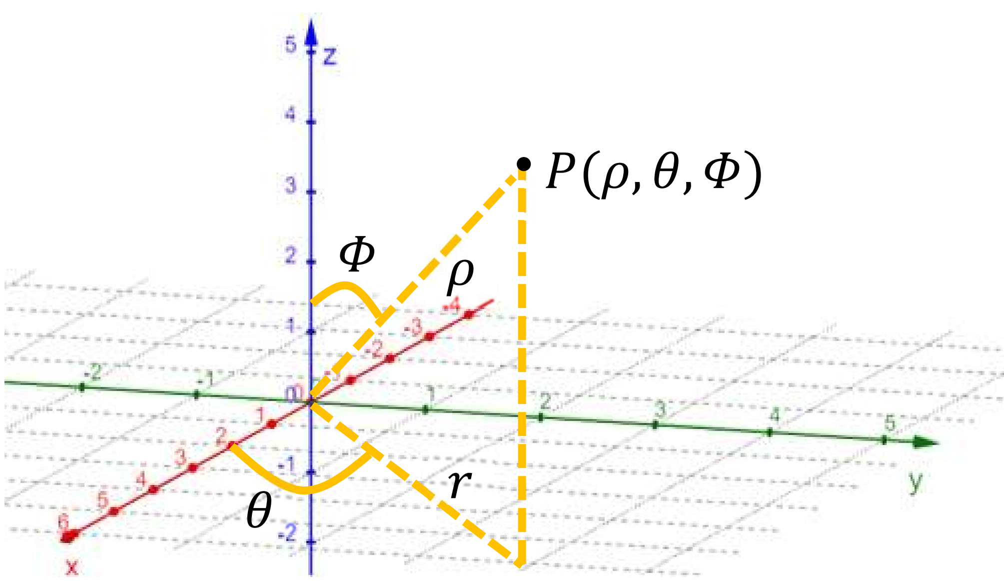
You can see that the shape will carve out a cylinder when rotated around the plane containing P, parallel to the (x,y) plane centered on the origin on the chart. **This is a LEVEL CURVE.**

FOR A SPHERE-ISH OBJECT

(Doesn't have to be a perfect sphere)

When your 3D regions are defined by cones and/or spheres, use spherical coordinates.

HOT TIPS: When you are setting up your equation, always set x=0 and z=0!



$$\begin{aligned} Where: \quad & r = \text{ the distance of the projected point to the origin} \\ & \Phi = \text{angle from the positive } z \\ & \rho = \text{the distance of } P \text{ from the origin} \\ & \quad = \sqrt{x^2 + y^2 + z^2} \quad and \\ & \rho(sin\theta) = r \end{aligned}$$

Like with cylinders, there is a specific order for working with spheres:

$$\int_{\theta=\alpha}^{\theta=\beta} \int_{\Phi=c}^{\Phi=d} \int_{\rho=}^{\rho=} f(x,y,z)\, \rho^2 sin\Phi \; dp \; d\Phi \; d\theta$$

Spherical Wedge formula mini-proof:

$$V = Arc\ length\ along\ \theta \cdot arc\ length\ along\ \Phi \cdot depth$$
$$\Delta V = \rho \sin \Phi \cdot \Delta\theta \cdot \rho \cdot \Delta\Phi \cdot \Delta\rho \ = \ \rho^2 \sin \Phi \Delta\rho\Delta\Phi\Delta\theta$$
$$dV = \rho^2 sin\Phi\ d\rho\ d\Phi\ d\theta$$

Jacobian Magic Trick

Want to make your impossible or really really really hard function disappear? You can reshape your axis to make an ellipse into a circle or a parallelogram into a square. Here's how:

For a double integral, the thing we integrate over is a region. So, for a "double substitution" we must change the region "R". By a transformation from $\mathbb{R} \in (x, y)$ to $\mathbb{H} \in (u, v)$ where the transformation is a 1-to-1 relationship between $\mathbb{R}$ and $\mathbb{H}$ and defines how (x, y) and (u, v) work together.

$$\int\int_R f(x,y)\, dA \quad \Rightarrow \quad \int\int f(g(u,v), h(u,v))\ J\ dudv$$

J stands for Jacobian. Sir Jacobian was a magician that died doing math in the 1400s. Just kidding, but can you imagine? Anyway, here's what a Jacobian does:

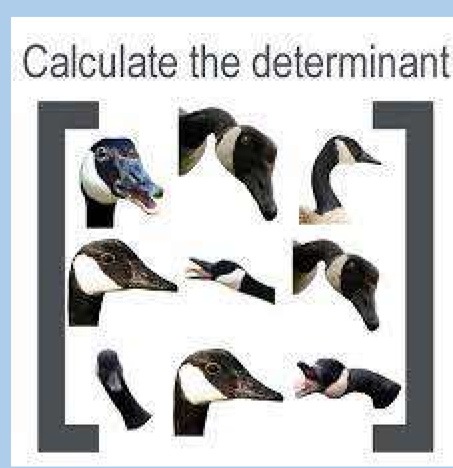
$$Jacobian = \frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2u & -1 \end{bmatrix} = -2 - 2u = -2(1+u)$$

(Find the determinant)

Example:

$$x = 2u + w, \qquad y = u^2 - v^2, \qquad z = u + v^2 - 2w^2$$

$$Jacobian = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 2u & -2v & 0 \\ 1 & 2v & -4w \end{bmatrix} \Rightarrow$$



$$Determinant = 2(8vw - 0) - 0(junk) + 1(4uv + 2v)$$

$$\therefore Jacobian = 16vw + 4uv + 2v$$

Now, what is that Jacobian thingy? It's the back office of a triple integral:

$$\iiint (Jacobian)\, du\, dv\, dw = \iiint (16vw + 4uv + 2v)\, du\, dv\, dw$$

Lebesgue (Horizontal) Integration

Ordinarily, this concept is introduced in topology courses. Once I learned it in topology, I was like, "Why didn't I learn this in Calc 3? It could have been super useful in understanding all that shit!" So, I will introduce it here.

All the integrals we have done so far have been Reimann integrals, which are hard to expand into higher dimensions without a crap-ton of work, and are dependent on continuity.

I have to pay a certain sum, which I have collected in my pocket. I can take the bills and coins out of my pocket and give them to the creditor in the order I find them until I have reached the total sum. This is the Reimann integral. But I can proceed differently. After I have taken all the money out of my pocket, I order the bills and coins according to identical values and then I pay the several heaps one after the other to the creditor. This is my integral.

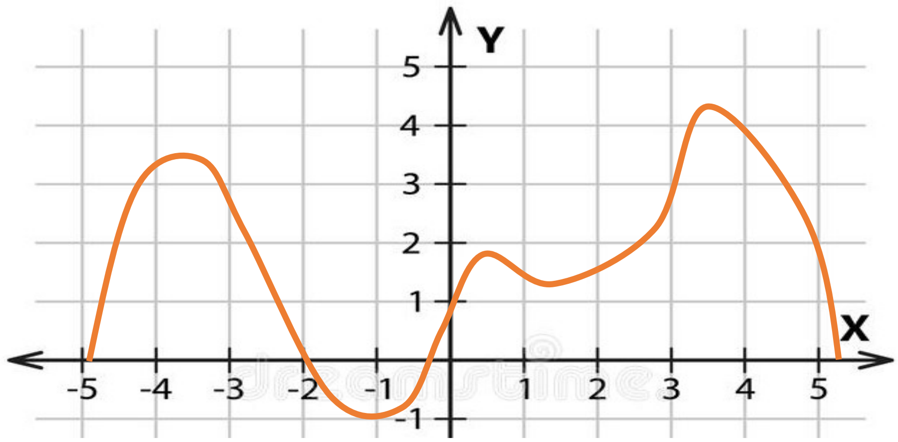
--Henri Lebesgue

WHY DO WE NEED THIS INTEGRAL?

In our daily lives, we don't deal with pure polynomials. Like for example, these things are not pure polynomials and sometimes cannot be integrated with Reimann techniques:

- Expected values
- Electrical currents
- Probability distributions
- Events that only take on certain and specific values
- Bayesian probability events
- Music (this would be a discrete function, since there are not infinite notes)

Consider how you would integrate this function horizontally:



We only need to consider a range of outputs here if we are doing a Lebesgue integral:

$$f(x), x \in [-1, 0, 1, 2, 3, 4, 5]$$

So, as another example, you could assign $A_0 \ when \ f(x) = 0$ if x is a rational number and $A_1 \ when \ f(x) = 1$ if x is an irrational number.

The horizontal integral would be written as:

$$\int_0^1 f(x) d\mu = 0 \cdot \mu(A_0) + 1 \cdot \mu(A_1)$$

(Whenever we see $d\mu$ we know it's a Lebesgue integral).

Lebesgue Integral of a Simple Function:

$$\int_a^b f(x) d\mu = \sum_{i=1}^{n} y_i \cdot \mu(A_{yi})$$

Lebesgue Integral of a Continuous Function:

$$\int_a^b f(x) d\mu = \lim_{n \to \infty} \int_a^b f_n(x) d\mu$$