Partial Derivatives and Gradient

You should also print my review sheet on Vector Calculus as well as Limits and Derivatives and Integrals Review. You may also want to print my trigonometry review sheet, as all of these will be helpful for multivariable calculus.

Domain and Range

To graph a function, you must have one more dimension than you have independent variables.

To graph the ***domain*** of a function, you must have the same number of dimensions as the number of independent variables. Each independent variable must have an axis.

Example:

$$f(x,y) = \frac{xy}{x-y} \;, \qquad x - y \neq 0 \qquad \therefore x \neq y$$

Domain: **the input** $D: \{(x,y) | x \neq y\}$ The domain is x and y such that x is not equal to y

Range: the output $R: \{z | -\infty < z < \infty\}$ The range (dependent variable) is z such that it's between negative and positive infinity.

Limits Along a Surface

Instead of approaching a point, these will be approaching a plane or higher-dimensional shape. Instead of having only 2 ways to get to the destination, there are infinite paths along a surface that approach a path.

$$\lim_{(x,y)\to(a,b)} f(x,y)$$

To prove this limit exists, we must prove that all paths approach the same point.

Option 1: Squeeze Theorem:

Also known as the **Sandwich Theorem**

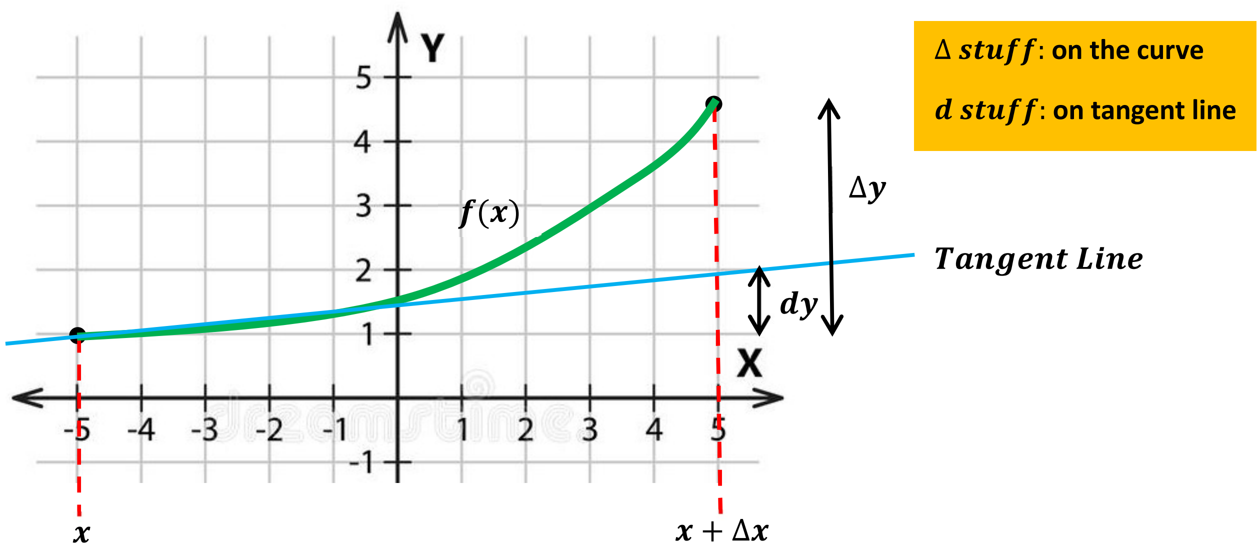
$$\begin{aligned}&If\ f(x) \leq g(x) \leq h(x)\ \forall\ numbers, and\\ &at\ some\ point\ x = k\ we\ have\ f(k) = h(k), then\\ &g(k) = f(k) = h(k)\end{aligned}$$

Option 2: Prove Continuity: If a function is continuous, the limits exist.

Option 3: Prove a Limit Does Not Exist: Show that along two paths, you get a different value as you approach the same point (x,y).

Introduction to Differentials

Most students were introduced to this vaguely in Calculus 1, however, probably not in sufficient time to understand the concept. Here's a refresher:



Δy: an increment that gives you the initial change in height from $x\ to\ \Delta x$

dy: a differential that gives the change in height from a point on the tangent line at x to another point on the tangent line at $x + \Delta x$.

As $\Delta x \to x$, then $\Delta y \approx dy$

Partial Derivatives

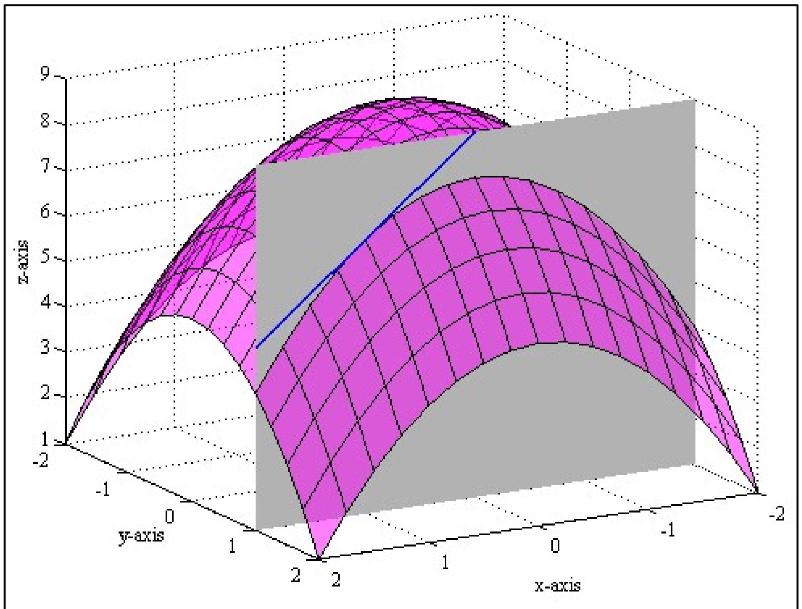
There are infinite tangents to a surface at a point. To find the slope of a tangent line to a surface at a point, you must restrict the tangent line to a direction.

Restricted along an axis: The plane must contain the axis as a subset, and the other independent variable's axis will be held constant.

Properties of Partial Derivatives: These are the same as what we learned on my Limits and Derivatives review.

In 2D, we have curves and tangent lines. And we will use derivatives.

In 3D, we have surfaces and tangent planes. And we will use differentials.



y = 1 tangent plane (in gray) is in the x-direction.

f_x = The blue line, which shows the slope of the tangent plane y=1 in the x-direction.

Partial Derivative Notation:

$$\frac{\partial f}{\partial x} = f_x: \quad holds\ y\ constant, so\ it\ gives\ the\ slope\ of\ the\ tangent\ line\ in\ the\ x\ direction$$

$$\frac{\partial f}{\partial y} = f_y: \quad holds\ x\ constant, so\ it\ gives\ the\ slope\ of\ the\ tangent\ line\ in\ the\ y\ direction$$

Example: $f(x) = 2x^2y^3 - 3x^2y + 2x^3 + 3y^2 + 1$

A lot of people try to use the product rule here. You don't need it.

$$\frac{\partial f}{\partial x} = 4xy^3 - 6xy + 6x^2 + 0 + 0$$

$$\frac{\partial f}{\partial y} = 6x^2y^2 - 3x^2 + 0 + 6y + 0$$

Total Differential: Take all the partial derivatives and add them up.

$$df(x, y, z) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = f_x dx + f_y dy + f_z dz$$

Chain Rule for Multivariable Functions: Must do a partial derivative.

$w = x^2 - y^2 \;\; ; \;\; x = t^2 + 1 \, , \qquad y = t^3 + t$

Step 1: Identify dependent, independent, and intermediary variables. In this case, w = dependent, t = independent, x & y are intermediaries

Step 2: Identify all potential paths and draw derivative tree

$\Delta t \rightarrow \Delta x \rightarrow \Delta w$

$\Delta t \rightarrow \Delta y \rightarrow \Delta w$

$t \; \left\{ \begin{matrix} x \\ y \end{matrix} \right. \; w$ → Any point where there is a branch in the tree (x and y), it's a partial derivative. Anywhere with only one path (t and w) is a derivative.

Step 3: Write equation incorporating all potential paths

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}$$

DIRECTIONAL DERIVATIVE: We calculate this so that we can base a derivative on a unit vector instead of some random line parallel to that unit derivative.

For a derivative to have a direction, it must have a unit vector on the plane of the independent variables.

Examples:

$\frac{\partial f}{\partial x}$ is a partial derivative in the direction of $\hat{\imath}$.

$\frac{\partial f}{\partial y}$ is a partial derivative in the direction of $\hat{\jmath}$.

GRADIENT: the vector for the rate of climb (grade) of a surface, where you can put your tangent line to get the steepest slope. It's also just the leftover part from dot product after separating the unit vector.

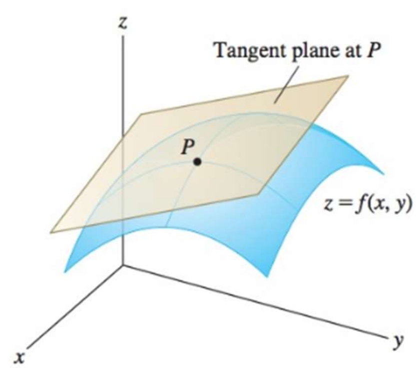
$$D\hat{u}f_{(x,y)} = f_x u_1 + f_y u_2 \quad \Rightarrow \quad D\hat{u}f_{(x,y)} = (f_x\hat{\imath} + f_y\hat{\jmath}) * \hat{u} \quad \therefore \quad (f_x\hat{\imath} + f_y\hat{\jmath}) = \nabla f_{(x,y)}$$

Properties of Gradient:

- Has one less dimension than the function.
- If $\nabla f = \vec{0}$, then $D\hat{u}f = 0 \quad \forall\, \hat{u}$
- $D\hat{u}f_{(x,y)}$ has its max value of $||\nabla f_{(x,y)}||$ and this will happen ONLY when $\hat{u} = C \cdot \nabla f$ when C is a scalar and $\hat{u}\ and\ C \cdot \nabla f$ are in the same direction.
- Any other θ give a number less than 1 and takes a fraction of ∇f. In other words, $D\hat{u}$ becomes less steep.
- $D\hat{u}f_{(x,y)}$ has its minimum value of $-||\nabla f_{(x,y)}||$ at $\theta = \pi$
- If $\hat{u}$ is not parallel to ∇f, think of $\hat{u}$ as turning $D\hat{u}f_{(x,y)}$ away from the direction of the steepest climb, ∇f.

NORMAL LINE: the line that is perpendicular (orthogonal) to the tangent line at the point of tangency

TANGENT PLANES: A slope of a tangent vector of the level curve.



- Always has one less dimension than the function of the surface it is tangent to
- Represented by a vector function:
  $\hat{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} \quad (Level\ curve\ at\ C)$
- $\nabla f_{(x,y)} \cdot \hat{r}'(t) = 0 \qquad \therefore$ **they're orthogonal, so the gradient gives the normal to a level curve or surface at a point**
- So, the function for the level curve is: $f\big(x(t), y(t)\big) = C$

Optimization:

The Holy Grail of Calculus 3

Want a super high-paying job using this stuff? Learn Optimization!

Relative Max: The highest point relative to other points – optimizes for the highest value of something, like revenues, area of property.

Relative Max for 1 Variable	Relative Max for 2 Variables
• Relative max is a peak of a curve • Can have more than one relative max • Given by critical points, $f'(x) = 0$ or where $f'(x)$ is undefined. • Critical points could be inflection points	• Relative max is a peak of a surface • Can have more than one relative max • Given by critical points, but BOTH f_x & f_y must be = 0 • Critical points can be undefined or saddle points

Relative Min: The lowest point relative to other points – optimizes for the lowest value of something, like distance travelled, time, or manufacturing costs.

Relative Min for 1 Variable	Relative Min for 2 Variables
• Relative min is a valley of a curve • Can have more than one relative min • Given by critical points, $f'(x) = 0$ or where $f'(x)$ is undefined. • Critical points could be inflection points	• Relative min is a bowl of a surface • Can have more than one relative min • Given by critical points, but BOTH f_x & f_y must be = 0 • Critical points can be undefined

Absolute Max: either at a relative max or at an endpoint. The highest relative max present in a function.

Absolute Min: either at a relative min or at an endpoint. The lowest relative min present in a function.

SECOND DERIVATIVE TEST:

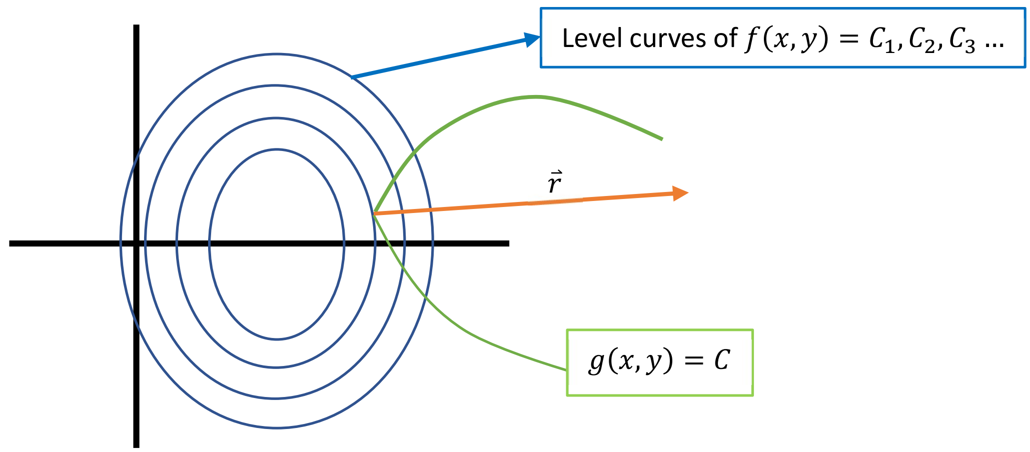
$D(x, y) = f_{xx}(x, y) \cdot f_{yy}(x, y) - (f_{xy})^2$

- $D(a, b) > 0, \quad f_{xx}(a, b) > 0 \quad \Rightarrow \quad (a, b)$ gives a relative min because the shape is concave up
- $D(a, b) > 0, \quad f_{xx}(a, b) < 0 \quad \Rightarrow \quad (a, b)$ gives a relative max because the shape is concave down
- $D(a, b) < 0, \quad \Rightarrow \quad (a, b)$ gives a saddle point because the derivatives are concave in opposite directions
- $D(a, b) = 0, \quad \Rightarrow$ test is inconclusive. How rude!

CONSTRAINED OPTIMIZATION: Given a function and a constraint.

All constrained maxima and minima are at critical points. If two curves intersect at one and only one point, their tangents are scalar multiples. The normal of their level curves are also scalar multiples.

Therefore: $\nabla f(x, y, z \dots) = \lambda \nabla g(x, y, z \dots)$



GRADIENT DESCENT: The fastest way down a surface function. Or a computer algorithm designed to find the absolute or relative minimum of a surface function.

You let $f(x, y, z \dots) = -g(x, y, z \dots)$ to find the gradient descent. (If you want the gradient ascent for maximization problems, just flip the signs!)

OPTIMIZING REPEAT PROCESSES: If you're given a sequence or repeating task, you might need to optimize it so that there is the smallest distance between the max and min values.

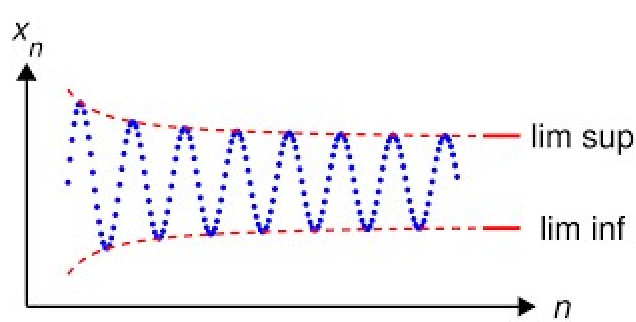
Okay so this is a small jump ahead into Real Analysis class, but it's pretty easy to understand. I'm not going to prove this. Save that for your Real Analysis curriculum.

Consider a repeat process, which has a range of values into which it falls.

Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of real numbers.

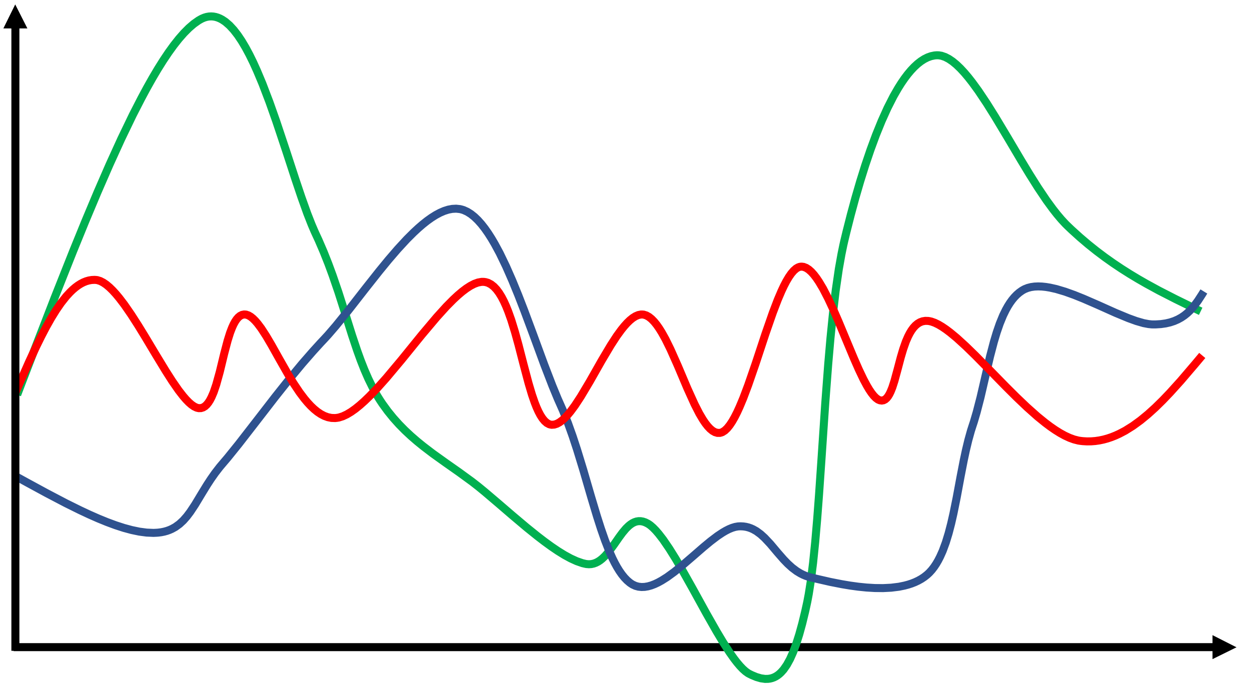
Limit Superior or **Limit Supremum:** $\limsup_{n \to \infty} a_n$ is where "a" is the largest accumulation value or cluster of $(a_n)_{n \in \mathbb{N}}$.

Limit Inferior or **Limit Infimum** $\liminf_{n \to \infty} a_n$ is where "a" is the smallest accumulation value or cluster of $(a_n)_{n \in \mathbb{N}}$.



So uh… WTF are these, why am I bringing them up, and how do we use them?

Consider this graph:



The red and blue lines are separate processes that go into making a product, and the green line represents your company's profit margin. You want to optimize your production so that the profit margin is the highest. You could do this task with liminf and limsup functions.

$$\therefore \sum \limsup_{n\to\infty} a_n \leftrightarrow \sum \liminf_{n\to\infty} a_n \;\notin \{\pm\infty\}$$