Vector Calculus in Colored Pencils

The most common way to make a mistake with vector calculus is by making a sign error. If you understand algebra and are familiar with derivatives and integrals, you can do vector calculus.

VECTORS are functions that have both a direction and a speed (magnitude). Direction is always shown on a coordinate system. ||Magnitude|| is given by the length of the vector. 3-D vectors are always shown on a right-hand coordinate system.

Unit vector = the direction part of a vector, denotated by a hat. Has a magnitude of 1.

- Along x-axis: $\hat{\imath}$
- Along y-axis: $\hat{\jmath}$
- Along z-axis: $\hat{k}$

Zero Vector $\vec{0}$ = <0,0,0> means the vector goes through the origin

When you use a scalar on a vector, it's like you're stretching the vector in a direction.

Dot product ALWAYS gives a scalar *as an answer because when you have two vectors, they will stretch on each other.*

Cross product ALWAYS gives a vector *as an answer.*

SCALAR = When you multiply a vector by a constant.

A scalar can only do 2 things to a vector:

1) Alter the magnitude of your vector or
2) Reverse the direction of your vector

Dot Product

Adds the product of corresponding components of vectors. It gives a scalar as an answer.

If:
$\vec{a} =< a_1, a_2, a_3 > \quad and \quad \vec{b} =< b_1, b_2, b_3 >$

$Then$:
$\vec{a} \cdot \vec{b} = (a_1)(\ b_1) + (a_2)(b_2) + (\ a_3)(b_3) = c$

$where\ c\ is\ a\ scalar.$

Properties:

1) $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$
   $Commutativity$

2) $\vec{v} \cdot (\vec{u} + \vec{w}) = \ \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{w}$
   $Associativity$

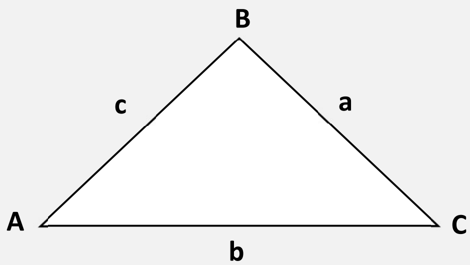
3) $C(\vec{v}) \cdot \vec{w} = C(\vec{v} \cdot \vec{w}) = \vec{v} \cdot (C\vec{w})$
   $Distribution\ Property$

4) $\vec{0} \cdot \vec{v} = 0$

5) $\vec{v} \cdot \vec{v} = \left|\left|v_1\right|\right|^2$

Remember the law of Cosines from Trigonometry?

$$a^2 + b^2 + c^2 - 2bccosA$$

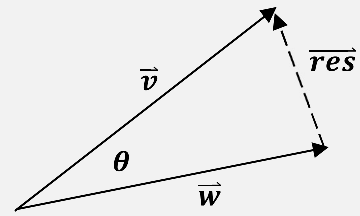


You can use it for dot product:

$$\vec{w} + \overrightarrow{res} = \vec{v} \ \therefore\ \overrightarrow{res} = \vec{v} - \vec{w}$$

$$\therefore \left|\left|\vec{v} - \vec{w}\right|\right|^2$$

$$= \left|\left|\vec{w}\right|\right|^2 + \left|\left|\vec{v}\right|\right|^2 - 2||\vec{w}||||\vec{v}||cos\theta$$



PARALLEL VECTORS: Two vectors are parallel if $\theta = 0 \ or \ \theta = \pi$

However, the easiest way to check if vectors are parallel is if they are scalar multiples. (Like, can you factor out a number? If so, they're parallel vectors.)

ORTHOGONAL = Means the vectors are perpendicular (but in 3d.)

For orthogonal vectors, $\theta = \frac{\pi}{2}$

$$\vec{v} \cdot \vec{w} = \left|\left|\vec{v}\right|\right| \cdot \left|\left|\vec{w}\right|\right| \cdot cos\frac{\pi}{2}, \quad \therefore\ \vec{v} \cdot \vec{w} = 0 \ if \ \vec{v} \perp \vec{w}$$

The unit vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$ are **mutually orthogonal.** This means that if you ever dot product two of them together, you will get 0.

Cross Product

Cross Product: Multiplies vectors to get a vector as an answer.

Remember from Algebra 2: Find the determinant of a matrix:

$$Determinant \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad + bc$$

Lots and lots and lots and lots of people make a sign error here.

$$Determinant \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} e & f \\ h & i \end{bmatrix} a - \begin{bmatrix} d & f \\ g & i \end{bmatrix} b + \begin{bmatrix} d & e \\ g & h \end{bmatrix} c$$

Properties:

With any two vectors, $\hat{a}\ and\ \hat{b}$,

1) $\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2)\hat{\imath} + (a_3b_1 - a_1b_3)\hat{\jmath} + (a_1b_2 - a_2b_1)\hat{k}$

2) $\vec{a} \times \vec{b}\ is\ orthogonal\ to\ both\ \hat{a}\ and\ \hat{b}$

3) $\vec{b} \times \vec{a}\ is\ also\ orthogonal\ to\ both\ \hat{a}\ and\ \hat{b}$

4) $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$ $\quad Meaning\ they're\ the\ same\ size, but\ in\ opposite\ directions.$

Vector Functions

Parametric Equation:

$$x = f(t), y = g(t), z = h(t)$$
$$where\ t\ is\ the\ parameter\ for\ some\ interaction\ on\ a\ common\ domain.$$

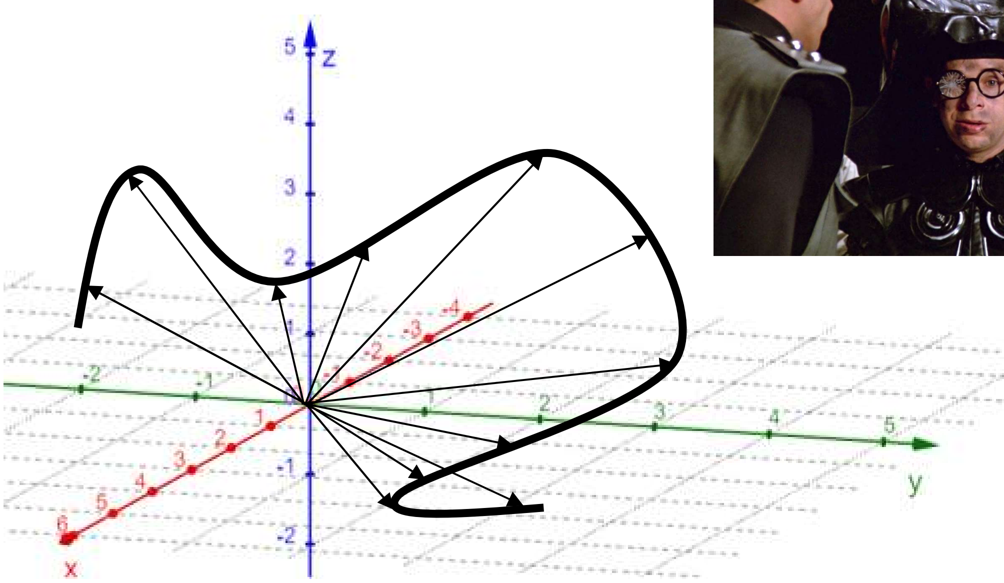
Vector Function: a parametrically defined function where the terminal points trace a curve in three dimensions. The fact that "t" has a certain domain gives $\hat{r}(t)$ an orientation.

Vectors are defined by $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$$\hat{r}(t) = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k} \rightarrow < f(t), g(t), h(t) >$$

The terminal points of the vectors create a curve through space (not a surface) for the entire domain of "t".

SPACE CURVE



Recall from Calculus 1: Definition of a Derivative:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Derivative of Vector = tangent vector to a space curve

$$\hat{r}'(t) = \lim_{\Delta t \to 0} \frac{\hat{r}(t+\Delta t) - \hat{r}(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}\hat{\imath} + \frac{g(t+\Delta t) - g(t)}{\Delta t}\hat{\jmath} + \frac{h(t+\Delta t) - h(t)}{\Delta t}\hat{k}$$

Example: Find the derivative.

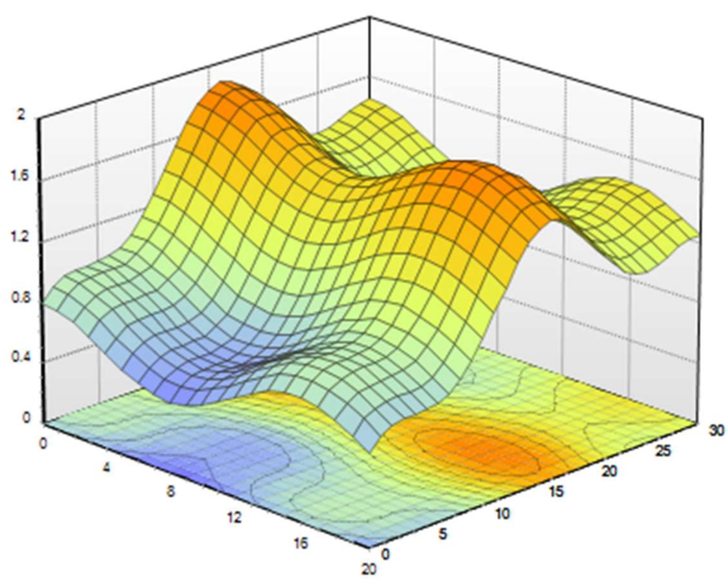
$$\hat{r}(t) = t\hat{\imath} + t^2\hat{\jmath} + t^4\hat{k} \qquad \rightarrow \qquad \hat{r}'(t) = \hat{\imath} + 2t\hat{\jmath} + 4t^3\hat{k}$$

Ok but what is the thing? It's a direction vector of a tangent line to the space curve at any "t" for our vector function.

Integral of Vector = area under a space curve. Note: your answer will be in the form of a vector. It can sometimes be shown as a shape with a shadow as below.

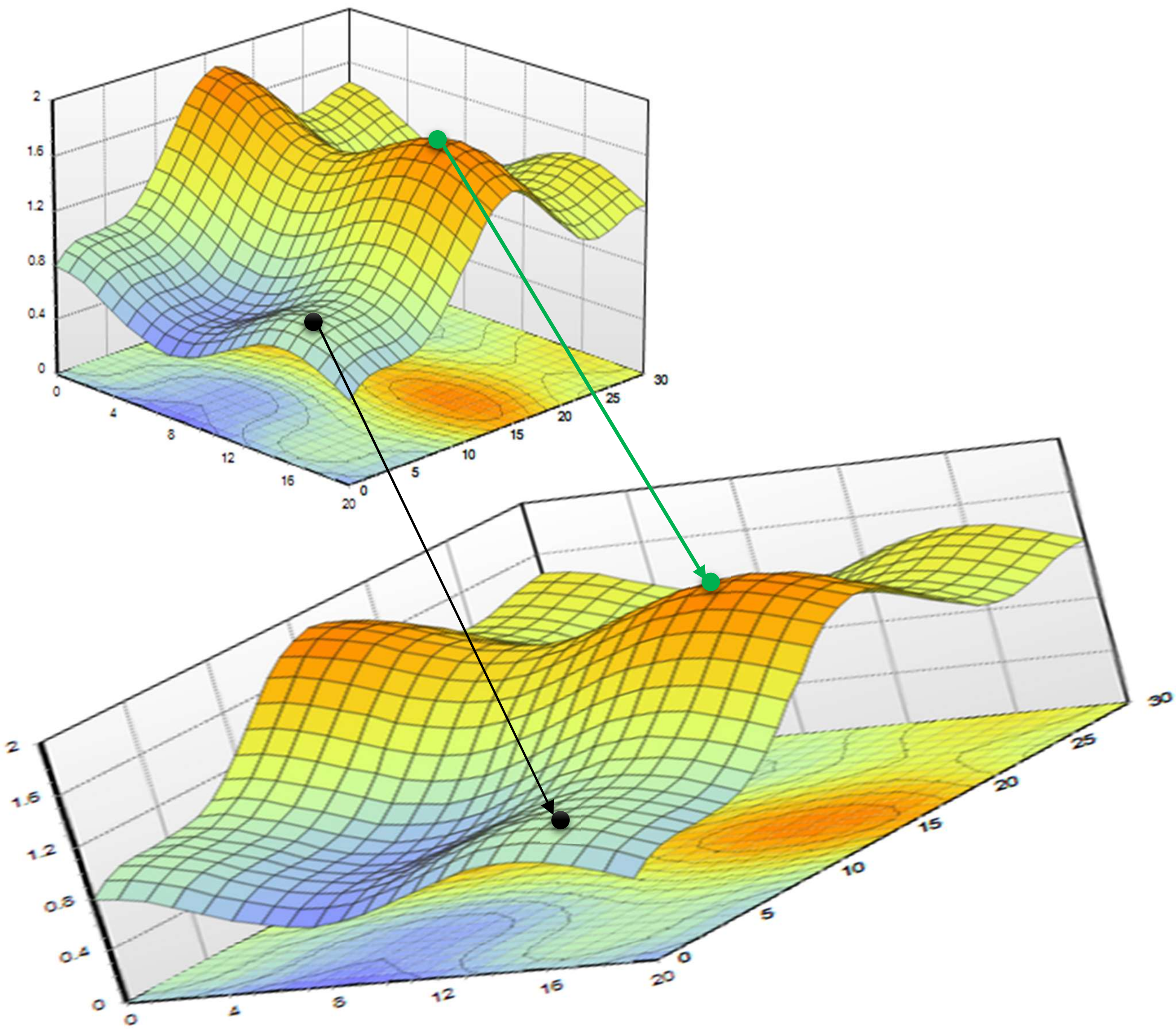
Example:

$$\int_0^1 \left(t\hat{\imath} + t^2\hat{\jmath} + t^4\hat{k}\right)dt \quad = \quad \frac{1}{2}t^2\hat{\imath} + \frac{1}{3}t^3\hat{\jmath} + \frac{1}{5}t^5\hat{k}\ \Big|_0^1 \quad = \quad \frac{1}{2}\hat{\imath} + \frac{1}{3}\hat{\jmath} + \frac{1}{5}\hat{k}$$



What the hell is the area under a space curve?

We can think about this as a sort of shadow. When a function is scaled, we can use the shadow to map the point to itself on a stretched and/or rotated coordinate system. Consider the following scales of the same function:



The shadow under the shape moved! You can use cross product on a Jacobian matrix to map black to black and green to green, or you can use integration to do the same process and arrive at the same answer.

Hopefully this helps you understand what the area under a space curve is, and why the integration of a vector returns a vector instead of a constant, a concept that was very confusing to me when I first took vector calculus.