1i)
$$2x^2 - 12x + 23$$
$$= 2\left[x^2 - 6x\right] + 23$$
$$= 2\left[(x-3)^2 - 9\right] + 23$$
$$= 2(x-3)^2 - 18 + 23$$
$$= 2(x-3)^2 + 5$$

ii)
$$2x^2 - 12x + 23 = 0$$
$$\Rightarrow 2(x-3)^2 + 5 = 0$$
$$\Rightarrow (x-3)^2 = -\frac{5}{2}$$
Cannot have square root of a negative $\therefore$ no real roots

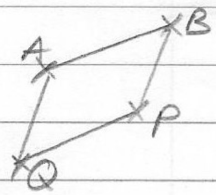
iii)
$$\underbrace{2x^2 - 12x + k = 0}$$
$a=2 \quad b=-12 \quad c=k$

$$D = b^2 - 4ac = (-12)^2 - 4(2)(k)$$
$$= 144 - 8k$$
Repeated roots $\therefore D = 0 \Rightarrow 144 - 8k = 0$
$$k = 18$$

2i)
$$\vec{AB} = \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \\ -3 \end{pmatrix}$$
$$AB = |\vec{AB}| = \sqrt{4^2 + 1^2 + 3^2}$$
$$= \sqrt{26}$$

ii)
$$\vec{OM} = \frac{1}{2}\left(\vec{OA} + \vec{OB}\right) = \frac{1}{2}\left(\begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} + \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} -2 \\ -3 \\ 7 \end{pmatrix} = \begin{pmatrix} -1 \\ -1.5 \\ 3.5 \end{pmatrix}$$

iii)
$$\vec{QP} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \\ -3 \end{pmatrix}$$




$\vec{AB} = \vec{QP}$ so AB and QP are parallel and equal in length $\therefore$ parallelogram

3i) Ayesha should spend 6 hours revising (graph is approx horizontal after $t=6$)
ii) Dave is unlikely to get zero marks, even with no revision

3iii) Bob thinks that, if he studies too long before the exam, he might get tired and his score might go down

iv) Chloe thinks that she will get the same score irrespective of how long she revises

4) $LHS = \sin^2(\theta+45) - \cos^2(\theta+45)$

$\boxed{\cancel{RHS = \sin 2\theta = 2\sin\theta\cos\theta}}$

$= [\sin(\theta+45)]^2 - [\cos(\theta+45)]^2$

$= [\sin\theta\cos45 + \cos\theta\sin45]^2 - [\cos\theta\cos45 - \sin\theta\sin45]^2$

$= \left[\frac{\sqrt{2}\sin\theta}{2} + \frac{\sqrt{2}\cos\theta}{2}\right]^2 - \left(\left[\frac{\sqrt{2}\cos\theta}{2} - \frac{\sqrt{2}\sin\theta}{2}\right]^2\right)$

$= \cancel{\frac{2\sin^2\theta}{4}} + \frac{\cancel{2}.\sqrt{2}\sin\theta.\sqrt{2}\cos\theta}{\cancel{2}\cdot 2} + \cancel{\frac{2\cos^2\theta}{4}} - \left(\cancel{\frac{2\cos^2\theta}{4}} - \frac{\cancel{2}.\sqrt{2}\cos\theta.\sqrt{2}\sin\theta}{\cancel{2}\cdot 2} + \cancel{\frac{2\sin^2\theta}{4}}\right)$

$= \sin\theta\cos\theta + \sin\theta\cos\theta$

$= 2\sin\theta\cos\theta$

$= \sin 2\theta = RHS$

5) i) Square numbers $= 1, 4, 9, 16, 25...$

$4 + 5 = 9$ (4 & 9 are square numbers, 5 is a prime)

Statement is false

ii)

| Charlie's proof | using $n^2=4$, $p=5$ & $m^2=9$. |
|---|---|
| $n^2 + p = m^2$ | $4 + 5 = 9$ ✓ |
| $\Rightarrow p = m^2 - n^2$ | $5 = 9 - 4$ ✓ |
| $\Rightarrow p = (m-n)(m+n)$ | $5 = (3-2)(3+2)$ ✓ |
| $\Rightarrow p =$ product of 2 integers | $5 = 1 \times 5$ ✓ |
| $\Rightarrow p =$ prime | $5 =$ prime × |

Charlie has shown $p =$ product of $(m-n)$ and $(m+n)$, however he overlooked the possibility that $(m-n)$ might equal 1

iii) Let square numbers $= n$ and $m$

and $853 = (m-n)(m+n) = 1 \times 853$

$\therefore$ $m - n = 1$ ①

$m + n = 853$ ②

①+② $2m = 854 \Rightarrow m = 427$

Sub in ① $427 - n = 1 \Rightarrow n = 426$

Ans $S = 426^2 = 181476$

Check

$181476 + 853 = 182329$

$427^2 = 182329$ ✓

6 i) $y = \frac{\ln x}{x}$

If $y=0$, $\frac{\ln x}{x} = 0$

$\ln x = 0$

$x = e^0 = 1$ //

ii) If $x=2$, $y = \frac{\ln 2}{2}$ $\therefore A = \left(2, \frac{\ln 2}{2}\right)$

If $x=4$, $y = \frac{\ln 4}{4} = \frac{\ln(2^2)}{4} = \frac{2\ln 2}{4} = \frac{\ln 2}{4}$ $\therefore B = \left(4, \frac{\ln 2}{2}\right)$

Grad (AB) $= \frac{\frac{\ln 2}{2} - \frac{\ln 2}{2}}{4-2} = 0$, so AB is parallel to x-axis //

iii) $\frac{dy}{dx} = \frac{x\left(\frac{1}{x}\right) - (\ln x)(1)}{x^2}$

$= \frac{1 - \ln x}{x^2}$

| $u = \ln x$ | $v = x$ |
|---|---|
| $\frac{du}{dx} = \frac{1}{x}$ | $\frac{dv}{dx} = 1$ |

If $\frac{dy}{dx} = 0$, $\frac{1-\ln x}{x^2} = 0$

$1 - \ln x = 0$

$\ln x = 1$

$x = e^1 = e$

If $x = e$, $y = \frac{\ln e}{e} = \frac{1}{e}$ $\therefore$ Turning point $= \left(e, \frac{1}{e}\right)$

iv) $\frac{d^2y}{dx^2} = \frac{\cancel{x^2}\left(-\frac{1}{\cancel{x}}\right) - [(1-\ln x)(2x)]}{(x^2)^2}$

$u = 1 - \ln x$ $\quad v = x^2$

$\frac{du}{dx} = -\frac{1}{x}$ $\quad \frac{dv}{dx} = 2x$

$= \frac{-x - [2x - 2x\ln x]}{x^4}$

$= \frac{-3x + 2x\ln x}{x^4}$ $\left(\text{or } \frac{-3 + 2\ln x}{x^3}\right)$

If $x = e$, $\frac{d^2y}{dx^2} = \frac{-3e + 2e\ln e}{e^4} = \frac{-e}{e^4} < 0$ $\therefore$ Max pt at $\left(e, \frac{1}{e}\right)$ //

7) Point C is intersection of $\begin{cases} y = 3 - 2x^2 & (1) \\ y = x & (2) \end{cases}$

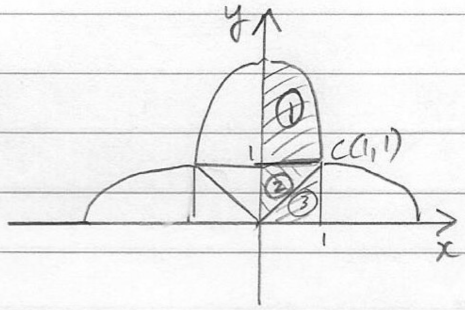
(Q7 contd)

Sub ② into ① $x = 3 - 2x^2$

$2x^2 + x - 3 = 0$

$(2x+3)(x-1) = 0$

$x = -\frac{3}{2}$ or $1$ $\quad \therefore C = (1,1)$




Area ② $= \frac{1}{2}bh = \frac{1}{2}(1)(1) = \frac{1}{2}$

Area ①+②+③ $= \int_0^1 (3 - 2x^2)\,dx$

$= \left[3x - \frac{2x^3}{3}\right]_0^1 = \left(3 - \frac{2}{3}\right) - (\cancel{0} - \cancel{0}) = \frac{7}{3}$

$\therefore$ Area ① $= \frac{7}{3} - 1 = \frac{4}{3}$

Total Area required = 8 × Area ① + 8 × Area ②

$= 8 \times \frac{4}{3} + 8 \times \frac{1}{2} = \frac{44}{3}$

## Section B: Statistics

8i) $X \sim N(20, 9)$

use calculator with $\sigma = 3$

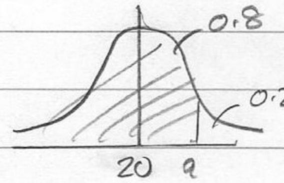
a) $P(X > 25) = 0.04779....$

$= 0.0478$ (3sf)

b) $P(X > a) = 0.2$

$a = 22.5248...$

$a = 22.5$ (3sf.)




use inv. normal on calculator with area=0.8, $\sigma=3$, $\mu=20$

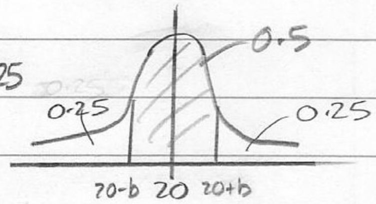
c) $P(20-b < X < 20+b) = 0.5$

By symmetry, $P(X<20-b) = P(X>20+b) = 0.25$

$20 - b = 17.9765$

$b = 2.02346...$

$b = 2.02$ (3sf.)

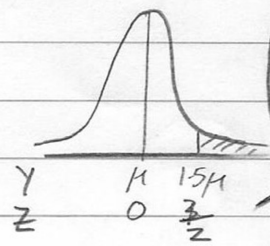



ii) $Y \sim N\left(\mu, \frac{\mu^2}{9}\right)$

$$P(Y > 1.5\mu) = P\left(Z > \frac{1.5\mu - \mu}{\left(\frac{\mu}{3}\right)}\right)$$

$$= P\left(Z > \frac{3}{2}\right)$$

$$= 0.0668 \quad (3s.f.)$$




Don't know $\mu$ or $\sigma$, so use $Z \sim N(0,1)$

9) Let $X$ = no of 2s (on 35 rolls)

$X \sim B(35, p)$

$H_0: p = \frac{1}{6}$

$H_1: p > \frac{1}{6}$ (1-tailed test, $\alpha = 4\%$)

Assuming $H_0$, $E(X) = (35)(\frac{1}{6}) = 5.833...$

$P(X \geq 10) = 1 - P(X \leq 9) = 1 - 0.9449 = 0.05500 = 5.5\%$.

$P(X \geq 11) = 1 - P(X \leq 10) = 1 - 0.9768 = 0.02319 = 2.32\% \longleftarrow$ (*)

$\therefore$ Rejection region (critical region) is <u>11 throws or more</u>

10) i) The trees are listed 1 to 784, so 810 does not correspond to any tree

ii) Not random because consecutive numbers have 1 digit in common

iii) $\overline{X}$ = sample mean of 50 trees

(10iii contd)

$\bar{X} \sim N\left(\mu, \frac{0.8^2}{50}\right)$

$H_0: \mu = 4.2$

$H_1: \mu < 4.2$ (1-tailed test, $\alpha = 2\%$)

Assuming $H_0$, $P(\bar{X} \leq 4.0) = 0.03854...$

$= 3.85\% > 2\%$.

Use calculator with $\sigma = 0.8/\sqrt{50}$

Not significant. Not enough evidence to reject $H_0$.

Not enough evidence to say the mean height of these trees is less than 4.2m

11 i)a) If the proportion, nationwide, of employees taking public transport is $p$, then proportion of employees driving is roughly $1-p$ (ignoring those walking etc.)

So if a local authority has population $P$, then $\begin{cases} x = \text{no taking public transp.} = Pp \\ y = \text{no driving} = P(1-p) \end{cases}$

If $p$ is roughly fixed, then both $x$ and $y$ are proportional to $P$, so this would give a strong positive correlation

FLAWED QUESTION the actual correlation (for 2011 data) is 24% – not a strong correlation!

b) Expect strong negative correlation: if a local authority has a high proportion of drivers, then it probably has a low proportion of those taking public transport.

ii a) The outlier would take the correlation closer to zero, e.g. if $r = -0.9$ without the borough, then it might become $r = -0.8$ with the borough

b) There are 348 authorities in total, so one outlier would have little effect

c) As a small proportion drive AND a small proportion use public transport, a lot of people must walk, ie. it must be a small borough.

FLAWED QUESTION People living in a borough don't have to work in the same borough, do they?

12 i)

| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $P(X=x)$ | $a$ | $\frac{1}{2}a$ | $\frac{1}{4}a$ | $\frac{1}{8}a$ | $\frac{1}{16}a$ |

$\sum P(X=x) = 1 \Rightarrow a + \frac{1}{2}a + \frac{1}{4}a + \frac{1}{8}a + \frac{1}{16}a = 1$

$(\times 16)$ $16a + 8a + 4a + 2a + a = 16$

$31a = 16 \Rightarrow a = \frac{16}{31}$

ii) $P(X = \text{odd}) = P(X = 1 \text{ or } 3 \text{ or } 5) = \frac{16}{31} + \frac{4}{31} + \frac{1}{31} = \frac{21}{31}$

iii) Let $S = X_1 + X_2$

$P(S \text{ odd}) = P(X_1 = \text{odd} \ \& \ X_2 = \text{even}) + P(X_1 = \text{even} \ \& \ X_2 = \text{odd})$

$= \frac{21}{31} \times \frac{10}{31} + \frac{10}{31} \times \frac{21}{31} = \frac{420}{961}$

iv) $P(S > 8 \mid S \text{ odd}) = \frac{P(S > 8 \cap S \text{ odd})}{P(S \text{ odd})}$

$= \frac{P(S = 9)}{P(S \text{ odd})}$

$= \frac{\left(\frac{2}{31} \times \frac{1}{31} + \frac{1}{31} \times \frac{2}{31}\right)}{\left(\frac{420}{961}\right)} = \frac{1}{105}$

v) Let $P(Y=1) = a$

then $P(Y=2) = \frac{1}{2}a$

$P(Y=3) = \frac{1}{4}a$ etc.

| $y$ | 1 | 2 | 3 | .... |
|---|---|---|---|---|
| $P(Y=y)$ | $a$ | $\frac{1}{2}a$ | $\frac{1}{4}a$ | .... |

$\sum P(Y=y) = 1 \Rightarrow \underbrace{a + \frac{1}{2}a + \frac{1}{4}a + \dots}_{} = 1$

GP $a = a, \ r = \frac{1}{2}, \ n = \infty$

$S_\infty = \frac{a}{1-r} \quad \therefore \quad \frac{a}{1-\frac{1}{2}} = 1$

$a = 1 \times \frac{1}{2} \quad \therefore P(Y=1) = \frac{1}{2}$

vi) X might be more appropriate, as Sheila won't keep trying indefinitely

FLAWED QUESTION
Assumes car will start eventually!

13i) $Y$ = no of late days (out of 450)

$Y \sim B(450, 0.15)$

If $P(Y > a) \approx \frac{1}{6}$

then $P(Y \leq a-1) = 1 - \frac{1}{6} = \frac{5}{6} = 0.8333\dots$

using trial-and-improvement,

$P(Y \leq 74) = 0.8229$

$P(Y \leq 75) = 0.8542$.

Use 'List' on the calculator

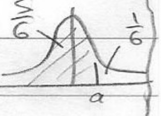
$\therefore a - 1 = 74 \Rightarrow a = 75$

Alternative

$Y \sim B(450, 0.15)$

$Y \approx N(67.5, 57.375)$

$P(Y > a) = \frac{1}{6}$




gives $a \approx 74.827\dots$

ie. $a = 75$ (nearest whole)

13ii) Require terms with $0.15^r$ and $0.15^{r+1}$

$\therefore$ treat 0.15 as the "$x$"

$$(0.15+0.85)^{50} = (0.85+0.15)^{50}$$
$$= 0.85^{50} + {}^{50}C_1 (0.85)^{49}(0.15)^1 + \dots + \underbrace{{}^{50}C_r (0.85)^{50-r}(0.15)^r}_{T_r} + \dots$$

So $T_r = {}^{50}C_r (0.85)^{50-r}(0.15)^r$

and $T_{r+1} = {}^{50}C_{r+1} (0.85)^{50-r-1}(0.15)^{r+1}$

Dividing, $\dfrac{T_r}{T_{r+1}} = \dfrac{{}^{50}C_r (0.85)^{50-r}(0.15)^r}{{}^{50}C_{r+1}(0.85)^{49-r}(0.15)^{r+1}}$ — ⊛

Now ${}^{50}C_r = \dfrac{50!}{r!(50-r)!}$ and ${}^{50}C_{r+1} = \dfrac{50!}{(r+1)!(50-r-1)!}$ ← see Formula book

$$\therefore \frac{{}^{50}C_r}{{}^{50}C_{r+1}} = \frac{\cancel{50!}}{r!(50-r)!} \times \frac{(r+1)!(49-r)!}{\cancel{50!}}$$

$$= \frac{[(r+1)\cancel{(r)}\cancel{(r-1)}\cancel{\dots}][\cancel{(49-r)}\cancel{(48-r)}\cancel{(47-r)}\cancel{\dots}]}{[\cancel{(r)}\cancel{(r-1)}\cancel{(r-2)}\cancel{\dots}][(50-r)\cancel{(49-r)}\cancel{(48-r)}\cancel{\dots}]}$$

$$= \frac{r+1}{50-r}$$ ← Check on calculator $\dfrac{{}^{50}C_{10}}{{}^{50}C_{11}}$ should equal $\dfrac{10+1}{50-10}$

Sub in ⊛

$$\frac{T_r}{T_{r+1}} = \left(\frac{r+1}{50-r}\right) \times \frac{\cancel{(0.85)^{50-r}}^{\,0.85}\ \cancel{(0.15)^r}}{\cancel{(0.85)^{49-r}}\ \cancel{(0.15)^{r+1}}_{\,0.15}}$$

$$= \left(\frac{r+1}{50-r}\right) \times \frac{0.85}{0.15} = \frac{17(r+1)}{3(50-r)}$$

iii)a) $X$ = no of late days (out of 50)

$X \sim B(50, 0.15)$

Using trial-and-improvement,

| $r$ | $P(X=r)$ | $P(X=r+1)$ |
|---|---|---|
| 6 | 0.1419 | 0.1574 ← ⊛ |
| 7 | 0.1574 | 0.1493 |

$P(X=r) \le P(X=r+1)$ for $r \le 6$

b) the most likely number of days = 7

Alternative for Q13iiia

Using part ii,

put $\dfrac{17(r+1)}{3(50-r)} \le 1$

$17r + 17 \le 150 - 3r$

$20r \le 133 \Rightarrow r \le 6.65$