1) $A=(1,5)$ $B=(4,17)$

$\text{Grad}(AB) = \frac{17-5}{4-1} = \frac{12}{3} = 4$ $\therefore$ perp grad $= \frac{-1}{4}$

Line through $(2,8)$ grad $= -\frac{1}{4}$

$y-8 = -\frac{1}{4}(x-2)$

$4y-32 = -x+2 \Rightarrow x+4y=34$

2)i)

x	0	0.5	1	1.5	2
y	1	1.284	2.718	9.488	54.598

$\int_0^2 e^{x^2}dx \approx \frac{1}{2}(0.5)\left\{(1+54.598)+2(1.284+2.718+9.488)\right\}$

$= \frac{1}{4}\left\{55.598 + 26.98\right\} = 20.6$ (3 s.f.)

ii) Use more strips

3) $x^4 - 5 = 4x^2$

$x^4 - 4x^2 - 5 = 0$ Let $X = x^2$

$X^2 - 4X - 5 = 0$ $X^2 = (x^2)^2 = x^4$

$(X-5)(X+1)=0$

$X=5$ or $X=-1$

$x^2=5$ or $\underbrace{x^2=-1}_{\text{no sol.}}$

$x = \pm\sqrt{5}$

4) Let $x = n^2+3n-1$

If n even, $n=2m$ $\therefore$ $x = (2m)^2 + 3(2m) - 1$

$= 4m^2 + 6m - 1$

$= 2(2m^2+3m)-1$ $\therefore x$ odd.

If n odd, $n=2m+1$ $\therefore$ $x = (2m+1)^2 + 3(2m+1) - 1$

$= 4m^2+4m+\cancel{1}+6m+3-\cancel{1} = 4m^2+10m+3$

$= 2(2m^2+5m+1)+1$ $\therefore x$ odd

In either case (n odd or n even), n^2+3n-1 will be odd

5) i) $x^2 + 6x + y^2 - 2y - 10 = 0$ — ①

$(x+3)^2 - 9 + (y-1)^2 - 1 - 10 = 0$

$(x+3)^2 + (y-1)^2 = 20$

Centre $= (-3, 1)$ radius $= \sqrt{20}$

ii) $y = 2x - 3$ — ②

Sub ② into ①

$x^2 + 6x + (2x-3)^2 - 2(2x-3) - 10 = 0.$

$x^2 + 6x + 4x^2 - 12x + 9 - 4x + 6 - 10 = 0.$

$5x^2 - 10x + 5 = 0$

$(\div 5)$ $x^2 - 2x + 1 = 0$

$(x-1)(x-1) = 0 \Rightarrow x = 1.$

Sub into ② $y = 2(1) - 3 = -1$

Ans $(x, y) = (1, -1)$

iii) Intersection at one point only $\therefore$ line is tangent at $(1,-1)$

6 i) $f(x) = 2x^3 - 7x^2 + 2x + 3.$

$(x-3)$ is a factor

$\therefore f(x) = (x-3)(2x^2 - x - 1)$

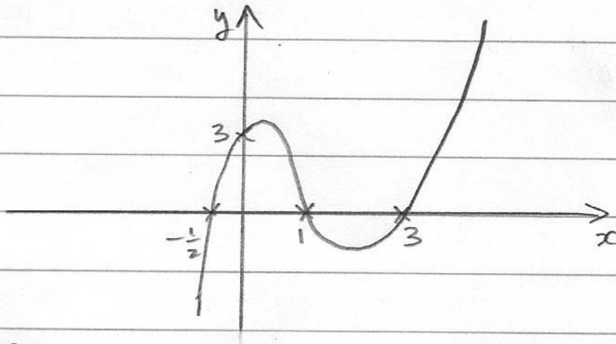
$= (x-3)(2x+1)(x-1)$

$$
\begin{array}{r}
2x^2 - x - 1 \\
x - 3 \overline{\smash{)}\, 2x^3 - 7x^2 + 2x + 3} \\
\underline{2x^3 - 6x^2} \\
-x^2 + 2x \\
\underline{-x^2 + 3x} \\
-x + 3 \\
\underline{-x + 3} \\
0.
\end{array}
$$

ii) If $x=0$, $y = 3$

If $y=0$, $(x-3)(2x+1)(x-1) = 0$

$x = 3, -\frac{1}{2}, 1.$



iii) If $f(x) < 0$, look for where graph is under the x-axis.

$\{x < -\frac{1}{2}\} \cup \{1 < x < 3\}$

iv) H. stretch $\times\frac{1}{2}$ takes $f(x) \to f(2x)$

ie. becomes $y = 2(2x)^3 - 7(2x)^2 + 2(2x) + 3$

$y = 16x^3 - 28x^2 + 4x + 3$

7i) Sequence = $\underbrace{150, 147, \ldots}_{}$

(GP) $a=150, \quad r=\frac{147}{150}=\frac{49}{50}=0.98$.

If $u_n = 120$

$ar^{n-1} = 120$

$150(0.98)^{n-1} = 120$.

$(0.98)^{n-1} = \frac{120}{150} = \frac{4}{5}$

$n-1 = \log_{0.98}\left(\frac{4}{5}\right)$

$n-1 = 11.045\ldots \Rightarrow n = 12.045\ldots$

If $n=12$, $u_{12} = 150(0.98)^{12-1} = 120.109\ldots$

If $n=13$, $u_{13} = 150(0.98)^{13-1} = 117.707\ldots$

$\therefore$ model predicts target achieved on 13^{TH} attempt

ii) If $S_n = 2974$

$\frac{a(1-r^n)}{1-r} = 2974$

$\frac{150(1-0.98^n)}{1-0.98} = 2974$

$1-0.98^n = \frac{2974(0.02)}{150}$

$1-0.98^n = 0.3965\ldots$

$0.98^n = 0.6034\ldots$

$n = \log_{0.98}(0.6034\ldots) = 24.9998\ldots$

$\therefore$ model predicts Chris has run 25 times

iii) In the short term, Chris may be unable to reduce his times each time.
In the long term, this model predicts times approaching to zero.

8i) $(4-x)^{-\frac{1}{2}} = 4^{-\frac{1}{2}}\left[1+\left(-\frac{x}{4}\right)\right]^{-\frac{1}{2}}$

$= 4^{-\frac{1}{2}}\left\{1 + \left(-\frac{1}{2}\right)\left(-\frac{x}{4}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{x^2}{16}\right) + \ldots\right\}$

$= \frac{1}{2}\left\{1 + \frac{x}{8} + \frac{3}{8}\cdot\frac{x^2}{16} + \ldots\right\} = \frac{1}{2} + \frac{x}{16} + \frac{3x^2}{256} + \ldots$

ii) $\frac{a+bx}{\sqrt{4-x}} = (a+bx)(4-x)^{-\frac{1}{2}} = (a+bx)\left(\frac{1}{2} + \frac{x}{16} + \frac{3x^2}{256} + \ldots\right)$

$= \frac{a}{2} + \frac{ax}{16} + \frac{3ax^2}{256} + \ldots$
$+ \frac{bx}{2} + \frac{bx^2}{16} + \ldots$

(Q8ii contd)

$$= \frac{a}{2} + \left(\frac{a}{16} + \frac{b}{2}\right)x + \left(\frac{3a}{256} + \frac{b}{16}\right)x^2 + \cdots = 16 - x + \cdots$$

Comparing, $\begin{cases} \frac{a}{12} = 16 & -① \\ \frac{a}{16} + \frac{b}{2} = -1 & -② \end{cases}$

From ①, $a = 32$

Sub in ② $\quad \frac{\cancel{32}^{2}}{\cancel{16}} + \frac{b}{2} = -1$

$$\frac{b}{2} = -3 \Rightarrow b = -6$$

9i) $f(x) = c + 8x - x^2$

$= -x^2 + 8x + c \quad \longleftarrow \cap$-shaped graph.

$= -[x^2 - 8x] + c$

$= -[(x-4)^2 - 16] + c$

$= -(x-4)^2 + 16 + c$

$\therefore$ maximum point at $(4, 16+c)$

If range is $f(x) \leq 19$ then $16 + c = 19 \Rightarrow c = 3$

ii) $ff(2) = f(f(2))$

$= f(c + 16 - 4)$

$= f(c + 12)$

$= c + 8(c+12) - [(c+12)^2]$

$= c + 8c + 96 - [c^2 + 24c + 144]$

$= 9c + 96 - c^2 - 24c - 144$

$= -c^2 - 15c - 48$

If $ff(2) = 8 \Rightarrow -c^2 - 15c - 48 = 8$

$c^2 + 15c + 56 = 0$

$(c+7)(c+8) = 0 \Rightarrow c = -7$ or -8

10i) $x = t + \frac{2}{t} = t + 2t^{-1}$ $-①$ $\quad y = t - \frac{2}{t} = t - 2t^{-1}$ $-②$

$$\underbrace{\frac{dx}{dt} = 1 - 2t^{-2} = 1 - \frac{2}{t^2} \qquad \frac{dy}{dt} = 1 + 2t^{-2} = 1 + \frac{2}{t^2}}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \left(1 + \frac{2}{t^2}\right) \times \frac{1}{\left(1 - \frac{2}{t^2}\right)}$$

(Q10i contd)

$$= \frac{\left[1 + \frac{2}{t^2}\right] \times t^2}{\left[1 - \frac{2}{t^2}\right] \times t^2} = \frac{t^2+2}{t^2-2}$$

ii) If $\frac{dy}{dx} = 0$, $\frac{t^2+2}{t^2-2} = 0$

$$t^2+2=0$$
$$t^2=-2$$

No solutions to $\frac{dy}{dx}=0$ $\therefore$ no stationary points

iii) $x+y = \left(t + \frac{2}{t}\right) + \left(t - \frac{2}{t}\right)$ using ① & ②

$$x+y = 2t \Rightarrow t = \frac{x+y}{2}$$

Sub in ① $x = \left(\frac{x+y}{2}\right) + \frac{2 \times 2}{\left(\frac{x+y}{2}\right) \times 2}$

$$x = \frac{x+y}{2} + \frac{4}{x+y}$$

$$x = \frac{(x+y)^2+8}{2(x+y)}$$

$$2x(x+y) = x^2+2xy+y^2+8$$
$$2x^2+\cancel{2xy} = x^2+\cancel{2xy}+y^2+8$$
$$x^2-y^2 = 8$$

11 i) $M = 300e^{-0.05t}$

If $t=0$, $M=300$ $\therefore$ initial value $=300$

If $M=150$, $300e^{-0.05t}=150$

$$e^{-0.05t} = \frac{1}{2}$$
$$-0.05t = \ln\left(\frac{1}{2}\right)$$
$$t = \frac{1}{-0.05}\ln\left(\frac{1}{2}\right) = 13.9 \text{ mins} \quad (3 s.f.)$$

ii) Let $M_2 = Ae^{-Bt}$

If $t=0$, $M_2=400$ so $Ae^0=400$

$$A=400$$

If $t=10$, $M_2=320$ so $400e^{-B(10)}=320$

$$e^{-10B} = \frac{\cancel{320}^{4}}{\cancel{400}_{5}}$$

(Q11ii contd)

$$-10B = \ln\left(\tfrac{4}{5}\right)$$

$$B = -\frac{1}{10}\ln\left(\frac{4}{5}\right) = 0.0223143\ldots$$
$$= 0.02231 \quad (4\text{sf})$$

$\therefore M_2 = 400e^{-0.02231t}$

Rate of change of $M_2 = \dfrac{dM_2}{dt} = -8.924e^{-0.02231t}$

Rate of change of original $M = \dfrac{dM_1}{dt} = -15e^{-0.05t}$

Let $T =$ time when $\dfrac{dM_1}{dt} = \dfrac{dM_2}{dt}$

$$\therefore \cancel{-}8.924e^{-0.02231T} = \cancel{-}15e^{-0.05T}$$

$$\frac{e^{-0.02231T}}{e^{-0.05T}} = \frac{15}{8.924}$$

$$e^{0.02769T} = 1.68086\ldots$$

$$0.02769T = \ln 1.68086\ldots$$

$$T = 18.75\ldots$$
$$= 18.8 \text{ mins} \quad (3\text{ sf})$$

12)

$$y = \frac{4\cos 2x}{3-\sin 2x}$$

$u = 4\cos 2x \qquad v = 3 - \sin 2x$

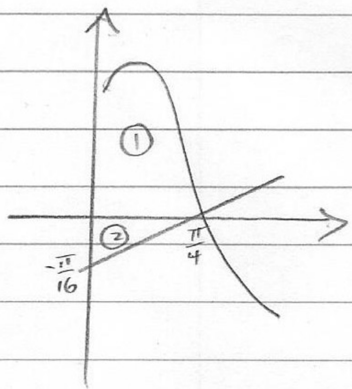
$\dfrac{du}{dx} = -8\sin 2x \qquad \dfrac{dv}{dx} = -2\cos 2x$

$$\frac{dy}{dx} = \frac{(3-\sin 2x)(-8\sin 2x) + (4\cos 2x)(\cancel{+}2\cos 2x)}{(3-\sin 2x)^2}$$

$$= \frac{-24\sin 2x + 8\sin^2 2x + 8\cos^2 2x}{(3-\sin 2x)^2}$$

At $\left(\frac{\pi}{4}, 0\right)$, $\dfrac{dy}{dx} = \dfrac{-24\sin\left(\frac{\pi}{2}\right) + 8\sin^2\left(\frac{\pi}{2}\right) \cancel{+ 8\cos^2\left(\frac{\pi}{2}\right)}}{\left(3-\sin\frac{\pi}{2}\right)^2}$

$$= \frac{-24+8}{2^2} = -4$$



Normal through $\left(\frac{\pi}{4}, 0\right)$ grad $= \frac{1}{4}$

$$y - 0 = \frac{1}{4}\left(x - \frac{\pi}{4}\right)$$

$$y = \frac{1}{4}x - \frac{\pi}{16}$$

Area of ② $= \frac{1}{2}\left(\frac{\pi}{4}\right)\left(\frac{\pi}{16}\right) = \frac{\pi^2}{128}$

(Q12 contd)

Area ① $= \displaystyle\int_{x=0}^{\frac{\pi}{4}} \left(\frac{4\cos 2x}{3-\sin 2x}\right) dx$

Let $u = 3-\sin 2x$

$\dfrac{du}{dx} = -2\cos 2x$

$\left(dx = \dfrac{du}{-2\cos 2x}\right)$

If $x=0$, $u=3-\sin 0=3$

If $x=\frac{\pi}{4}$, $u=3-\sin\frac{\pi}{2}=2$

$= \displaystyle\int \frac{\cancel{4}^{2}\cancel{\cos 2x}}{u} \cdot \frac{du}{(\cancel{-2\cos 2x})}$

$= -2\displaystyle\int_{u=3}^{u=2} \frac{1}{u}\, du$

$= -2\left[\ln u\right]_{u=3}^{u=2}$

$= -2\left\{\ln 2 - \ln 3\right\} = -2\ln\left(\frac{2}{3}\right) = 2\ln\left(\frac{3}{2}\right)$ {OR $\ln\frac{9}{4}$}

$\therefore$ Total area required $= \dfrac{\pi^2}{128} + 2\ln\left(\dfrac{3}{2}\right)$ //

13 i) $\dfrac{dN}{dt} = \dfrac{k}{N}$ //

ii) $\displaystyle\int N\, dN = \int k\, dt$

$\dfrac{N^2}{2} = kt + C$

(GS) $N^2 = 2kt + A$.

If $t=0$, $N=400$ so $400^2 = 2\cancel{k(0)} + A \Rightarrow A = 160\,000$

If $t=1$, $N=440$ so $440^2 = 2k(1) + 160000$

$2k = 33600 \Rightarrow k = 16800$

$N^2 = 33600t + 160000$

(PS) $N = \sqrt{160000 + 33600t}$ //

iii) $\dfrac{dN}{dt} = \dfrac{N^2}{3988e^{0.2t}}$

$\displaystyle\int \frac{dN}{N^2} = \int \frac{dt}{3988e^{0.2t}}$

$\displaystyle\int N^{-2}\, dN = \int \frac{e^{-0.2t}}{3988}\, dt$

$-N^{-1} = \dfrac{e^{-0.2t}}{3988(-0.2)} + C$

(Q13ii contd) (GS) $-\frac{1}{N} = -\frac{5}{3988e^{0.2t}} + C$

If $t=0$, $N=400$ so $-\frac{1}{400} = -\frac{5}{3988(1)} + C$

$C = -\frac{497}{398800}$

$+\frac{1}{N} = +\frac{5}{3988e^{0.2t}} + \frac{497}{398800}$

$\frac{1}{N} = \frac{500+497e^{0.2t}}{398800e^{0.2t}}$

(PS) $N = \frac{398800e^{0.2t}}{500+497e^{0.2t}}$

[Check: If $t=1$, $N = \frac{398800e^{0.2}}{500+497e^{0.2}} \approx 440$]

zii) 1st model

$N = \sqrt{160\,000 + 33600t}$

t	10	100	1000	1 000 000
N	704	1876	5810	183303

As $t \to \infty$, $N \to \infty$ (Population tends to increase indefinitely)

2nd model

$N = \frac{(398800e^{0.2t}) \times e^{-0.2t}}{(500+497e^{0.2t}) \times e^{-0.2t}}$

t	10	100	1000
N	706.25	802.41	802.41

$= \frac{398800}{500e^{-0.2t} + 497}$

As $t \to \infty$, $e^{-0.2t} \to 0$

$\therefore N \to \frac{398800}{497} = 802$ insects

(Populations tends towards limit of 802)