1a) i) Let $y = \dfrac{x^2}{2x+1}$

$u = x^2 \qquad v = 2x+1$

$\dfrac{du}{dx} = 2x \qquad \dfrac{dv}{dx} = 2$

$$\frac{dy}{dx} = \frac{(2x+1)(2x) - (x^2)(2)}{(2x+1)^2}$$

$$= \frac{4x^2 + 2x - 2x^2}{(2x+1)^2} = \frac{2x^2+2x}{(2x+1)^2}$$

[OR $\dfrac{2x(x+1)}{(2x+1)^2}$]

ii) Let $y = \tan(x^2 - 3x)$

$$\frac{dy}{dx} = \sec^2(x^2-3x) \times (2x-3)$$

$$= (2x-3)\sec^2(x^2-3x)$$

b) $\displaystyle\int \frac{1}{\sqrt{x}-1}\,dx$

Let $u = \sqrt{x} - 1 = x^{\frac{1}{2}} - 1$

$\dfrac{du}{dx} = \dfrac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2x^{\frac{1}{2}}}$

$(dx = 2x^{\frac{1}{2}}\,du)$

$x^{\frac{1}{2}} = u+1$

$2x^{\frac{1}{2}} = (2u+2)$

$$= \int \frac{1}{u} \cdot 2x^{\frac{1}{2}}\,du$$

$$= \int \frac{(2u+2)}{u}\,du$$

$$= \int \left(2 + \frac{2}{u}\right) du = 2u + 2\ln u + c$$

$$= 2x^{\frac{1}{2}} - 2 + 2\ln\left(x^{\frac{1}{2}} - 1\right) + c$$

[OR $2\sqrt{x} + 2\ln(\sqrt{x}-1) + c$]

c) $\displaystyle\int \frac{x-2}{2x^2-8x-1}\,dx$

Let $u = 2x^2 - 8x - 1$

$\dfrac{du}{dx} = 4x - 8$

$\left(dx = \dfrac{du}{4(x-2)}\right)$

$$= \int \frac{\cancel{x-2}}{u} \cdot \frac{du}{4\cancel{(x-2)}}$$

$$= \frac{1}{4}\int \frac{1}{u}\,du = \frac{1}{4}\ln u + c$$

$$= \frac{1}{4}\ln\left(2x^2 - 8x - 1\right) + c$$

[OR $\dfrac{1}{4}\ln\left(8x^2 - 32x - 4\right) + c$]

2a) $(3-2x)^8 = \left[\boxed{3} + (-2x)\right]^8$

$$= 3^8 + {}^8C_1(3)^7(-2x) + {}^8C_2(3)^6(-2x)^2 + {}^8C_3(3)^5(-2x)^3 + {}^8C_4(3)^4(-2x)^4 + {}^8C_5(3)^3(-2x)^5 + \dots$$

Term with $x^5 = {}^8C_5(3)^3(-2x)^5$

$$= 56 \times 27 \times -32x^5 = -48384x^5$$

coeff of $x^5 = -48384$

2bi) $(1+3x)^{0.5} = (1+(3x))^{0.5}$

$$= 1 + (0.5)(3x) + \frac{(0.5)(-0.5)}{2}(9x^2) + \frac{(0.5)(-0.5)(-1.5)}{6}(27x^3) + \ldots$$

$$= 1 + \frac{3x}{2} - \frac{9x^2}{8} + \frac{27}{16}x^3 + \ldots \quad (*)$$

ii) Valid for $|3x| < 1$, ie. $-\frac{1}{3} < x < \frac{1}{3}$

iii) To use $(1+3x)^{0.5}$ to get a value for $\sqrt{103}$, need to put $x = 0.01$ so that

$$(1+3x)^{0.5} = (1+0.03)^{0.5}$$

$$= (1.03)^{0.5} = \sqrt{\frac{103}{100}} = \frac{1}{10}\sqrt{103}$$

Sub $x = 0.01$ in $(*)$ $\frac{1}{10}\sqrt{103} \approx 1 + \frac{3(0.01)}{2} - \frac{9(0.01)^2}{8} + \frac{27}{16}(0.01)^3 + \ldots$

$$= 1.014889188$$

so $\sqrt{103} \approx 10.14889$ (5 dp)

In fact $\sqrt{103} = 10.14889157$ (from calculator)

3a)i) $x = 3 + 2\cos\theta$ $\qquad y = -4 + 2\sin\theta$

$\cos\theta = \frac{x-3}{2}$ $\qquad \sin\theta = \frac{y+4}{2}$

Using $\cos^2\theta + \sin^2\theta = 1$ then $\left(\frac{x-3}{2}\right)^2 + \left(\frac{y+4}{2}\right)^2 = 1$

$$\frac{(x-3)^2}{4} + \frac{(y+4)^2}{4} = 1$$

$$(x-3)^2 + (y+4)^2 = 4$$

ii) Centre $= (3,-4)$, radius $= 2$

b) $x = 4\cos t$ $\qquad y = 2\sin t$

$\frac{dx}{dt} = -4\sin t$ $\qquad \frac{dy}{dt} = 2\cos t$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 2\cos t \times \frac{1}{-4\sin t} = -\frac{\cos t}{2\sin t}.$$

If $t = \frac{\pi}{6}$ then $\begin{cases} \frac{dy}{dx} = -\frac{\cos\frac{\pi}{6}}{2\sin\frac{\pi}{6}} = \frac{-\sqrt{3}}{2} \\ x = 4\cos\frac{\pi}{6} = 2\sqrt{3} \\ y = 2\sin\frac{\pi}{6} = 1 \end{cases}$

Tangent thru $(2\sqrt{3}, 1)$ grad $= -\frac{\sqrt{3}}{2}$

$$y - 1 = -\frac{\sqrt{3}}{2}(x - 2\sqrt{3})$$

$$2y - 2 = -\sqrt{3}x + 6$$

$$\sqrt{3}x + 2y - 8 = 0.$$

Cuts x-axis where $y=0$ ie. $\sqrt{3}x - 8 = 0$ ~~$\Rightarrow x=$~~

$x = \frac{8\sqrt{3}}{3}$ Ans $= \left(\frac{8\sqrt{3}}{3}, 0\right)$

4a) Rate of growth = Gradient

In model A, the rate of growth will be slow at the start and the end, faster in the middle $(7 < t < 17)$. In model B, the rate of growth starts larger and decreases as t increases.

b) i) $P_A = 20 + 1000e^{-\frac{(t-20)^2}{100}}$

For $t > 20$, $(t-20)^2$ will grow

so $e^{-\frac{(t-20)^2}{100}}$ will decrease

and P will decrease, gradually approaching 20

ii) $P_B = 20 + 1000(1 - e^{-\frac{t}{5}})$

For $t > 20$, $e^{-\frac{t}{5}}$ will decrease, getting gradually closer to zero

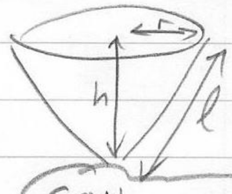
so $1000(1 - e^{-\frac{t}{5}})$ will get closer to 1000

and P will gradually approach 1020

c) i) model A suggests food supply will be mostly used up (only small population of animals can be supported)

model B suggests food supply will stabilise

5a) Inverted cone with no lid.



GOAL

$V = \frac{1}{3}\pi\sqrt{16r^2 - r^6}$

Area $S = \cancel{\pi} rl = 4\cancel{\pi}$

$\therefore rl = 4$ —①

Volume $V = \frac{1}{3}\pi r^2 h$ —②

By Pythagoras, $l^2 = r^2 + h^2$

$l = \sqrt{r^2 + h^2}$

Rewrite ① $r\sqrt{r^2 + h^2} = 4$

$r^2(r^2 + h^2) = 16$

$r^4 + r^2h^2 = 16$

$r^2h^2 = 16 - r^4 \Rightarrow h^2 = \frac{16 - r^4}{r^2} \Rightarrow h = \frac{\sqrt{16 - r^4}}{r}$

Sub in ② $V = \frac{1}{3}\pi \cancel{r^2}\left(\frac{\sqrt{16-r^4}}{\cancel{r}}\right)$

$= \frac{1}{3}\pi\sqrt{16r^2-r^6}$ //

5b) $\boxed{V = \frac{1}{3}\pi(16r^2-r^6)^{\frac{1}{2}}}$

$\frac{dV}{dr} = \frac{1}{6}\pi\left(16r^2-r^6\right)^{-\frac{1}{2}} \cdot \left(32r-6r^5\right)$

$= \frac{1}{\cancel{6}_3}\pi\,\frac{(\cancel{2}r)(16-3r^4)}{(16r^2-r^6)^{\frac{1}{2}}} = \frac{\pi\cancel{r}(16-3r^4)}{3\cancel{r}(16-r^4)^{\frac{1}{2}}}$

If $\frac{dV}{dr}=0$ then $\frac{\pi(16-3r^4)}{3(16-r^4)^{\frac{1}{2}}} = 0$

$16-3r^4 = 0$

$r^4 = \frac{16}{3} \Rightarrow r = \sqrt[4]{\frac{16}{3}} = 1.51967\ldots$

So $V_{max} = \frac{1}{3}\pi\left(16(1.51967\ldots)^2 - (1.51967\ldots)^6\right)^{\frac{1}{2}}$

$= 5.1974\ldots = 5.20$ // (3sf)

6) Assume n is even and $n \geq 4$
and that 2^n-1 is prime.
Then let $n=2m$ with $m \geq 2$

$\therefore 2^n-1 = 2^{2m}-1$

$= (2^m)^2-1$

$= (2^m+1)(2^m-1)$ ⊛

n	2^n-1	
4	15	not prime
6	63	not prime
8	255	not prime
10	1023	not prime
12	4095	not prime

Because $m \geq 2$, then $2^m \geq 4$
so $(2^m+1) \geq 5$ and $(2^m-1) \geq 3$

So ⊛ is not prime: CONTRADICTION.
Therefore, if n is an even integer >2 then 2^n-1 is not prime //

7) $36x^4 + 12x^3 + 7x^2 + x - 2 = 0$ $u = 6x^2 + x$

$(u^2 - \cancel{12x^3} - x^2) + \cancel{12x^3} + 7x^2 + x - 2 = 0$ $u^2 = (6x^2+x)^2$

$u^2 + 6x^2 + x - 2 = 0$ $= 36x^4 + 12x^3 + x^2$

$u^2 + u - 2 = 0$

$$(u+2)(u-1)=0$$

$u=-2$ or $u=1$

$6x^2+x=-2$ $\qquad$ $6x^2+x=1$

$6x^2+x+2=0$ $\qquad$ $6x^2+x-1=0$

$\cancel{(3x-2)(2x-1)=0}$ $\qquad$ $(3x-1)(2x+1)=0$

$x=\dfrac{-1\pm\sqrt{1-4(6)(2)}}{12}$ $\qquad$ $x=\frac{1}{3}$ or $x=-\frac{1}{2}$.

(no sols) $\qquad$ Ans: $x=\frac{1}{3}$ or $-\frac{1}{2}$ //

Section B: Statistics

8a) $M=\left(\dfrac{17+1}{2}\right)^{th}=9^{th}=65$ cm //

$Q_1=\left(\dfrac{17+1}{4}\right)^{th}=4.5^{th}=\frac{1}{2}(4^{th}+5^{th})=\frac{1}{2}(59+63)=61$

$Q_3=3\left(\dfrac{17+1}{4}\right)^{th}=13.5^{th}=\frac{1}{2}(13^{th}+14^{th})=\frac{1}{2}(75+77)=76$

$IQR=Q_3-Q_1=76-61=15$ cm //

b) mean $=\dfrac{\Sigma x}{n}=\dfrac{55+\ldots+99}{17}=\dfrac{1173}{17}=69$ cm //

Variance $=\dfrac{\Sigma x^2}{n}-\bar{x}^2=\dfrac{55^2+\ldots+99^2}{17}-(69)^2$

$=\dfrac{82811}{17}-(69)^2=110.235\ldots$

s.d. $=\sqrt{\text{Var}}=10.5$ cm //

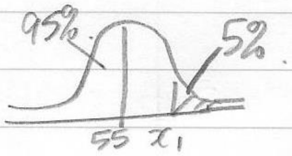
c) An advantage of the median is that it is not affected by outliers (in this case, the 99)

9a) Let X = mass of a plum

$X\sim N(55,\sqrt{18}^2)$

$P(50.0<X<60.0)=0.761$ // (3sf)

b) If $P(X<x_1)=0.05$



$x_1=61.9785\ldots$

$x_1=62.0$ // (3sf)

9a iii) Let $\bar{X}$ = sample mean of 10 plums

$\bar{X} \sim N\left(55, \frac{18}{10}\right)$

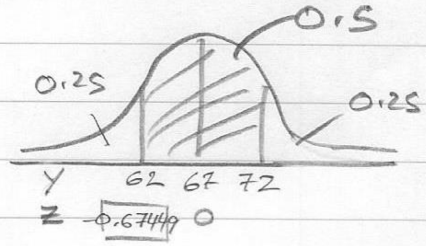
P(total weight < 530g) = $P(\bar{X} < 53)$

$= 0.0680$ (3sf)

b) Let Y = weight of an apple

$Y \sim N(67, \sigma^2)$

$P(62 < Y < 72) = 0.5$



From sketch, diagram is symmetric

$Z = \frac{X-\mu}{\sigma} \Rightarrow -0.67449 = \frac{62-67}{\sigma}$

$\sigma = 7.41$ (3sf)

10) Let $\bar{X}$ = sample mean

$\bar{X} \sim N\left(\mu, \frac{0.000\,0409}{50}\right)$

$H_0: \mu = 0.0340$

$H_1: \mu \neq 0.0340$ (2-tailed test, sig = 5%, 2.5% each tail)

$P(\bar{X} < 0.0325) = 0.0486$

$= 4.86\% > 2.5\%$.

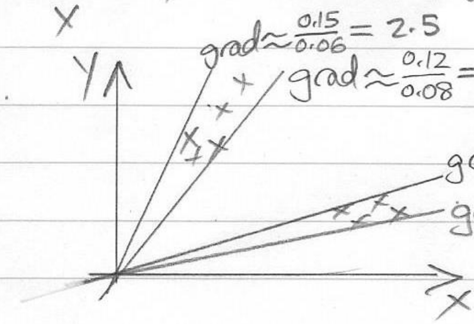
NOT SIGNIFICANT. Not enough evidence to reject H_0.

Not enough evidence to say mean level of pollutant has changed.

11a) If $k = \frac{P_{young}}{P_{senior}}$ and $P_{young} = Y$, $P_{senior} = X$

then $k = \frac{Y}{X} \Rightarrow Y = kX$ (Straight line through origin.

From diagram.



So Ranges chosen: $0.2 < k < 0.24$ and $1.5 < k < 2.5$

11 b i) If r = PMCC and ρ = population correlation

$H_0: \rho = 0$

$H_1: \rho < 0$ (1-tailed test, sig = 1%)

Test statistic $= r = -0.797$

Critical value $= -0.5577$

$0.797 > 0.5577$ $\therefore$ significant

ii) This result is unreliable, as selection method has systematically removed the Local Authorities 'in the middle' to leave two clusters

c) High proportion of people aged 65 and over

d) Universities would be interested in Local Authorities with a high proportion of people aged 18-24, ie. the points in top left corner of the diagram

12 a)

x	1	2	3	4	5
$P(X=x)$	k	$2k$	$3k$	$4k$	$5k$

$\sum P(X=x) = 1 \quad \therefore \quad k + 2k + 3k + 4k + 5k = 1$

$15k = 1 \Rightarrow k = \frac{1}{15}$

$P(X=3) = 3\left(\frac{1}{15}\right) = 0.2$

b)

x	1	2	3	4	5
$P(X=x)$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15} = \frac{1}{5}$	$\frac{4}{15}$	$\frac{5}{15} = \frac{1}{3}$

c) i) $T = X_1 + X_2$

$P(T=7) = P(X_1=5, X_2=2) + P(X_1=4, X_2=3) \ldots \ldots P(X_1=2, X_2=5)$

$= \frac{5}{15} \times \frac{2}{15} + \frac{4}{15} \times \frac{3}{15} + \ldots + \frac{2}{15} \times \frac{5}{15}$

$= \frac{10 + 12 + 12 + 10}{15^2} = \frac{44}{225}$ (or 0.1955...)

ii) $P(X_1 = 2 \text{ or } X_2 = 2 \mid T=7) = \frac{P((X_1=2 \text{ or } X_2=2) \cap T=7)}{P(T=7)}$

$= \frac{P(X_1=2 \text{ and } T=7) + P(X_2=2 \text{ and } T=7)}{P(T=7)}$

$= \frac{\left(\frac{2}{15} \times \frac{5}{15}\right) + \left(\frac{5}{15} \times \frac{2}{15}\right)}{\left(\frac{44}{225}\right)} = \frac{5}{11}$ (or 0.4545...)

13) X = no of adults using app (out of 5000)

$X \sim B(5000, 0.26)$

$\ldots \; n-1 \; n \; n+1 \ldots$

Solve $P(X<n) < 0.025$. ie. $P(X \leq n-1) < 0.025$.

Method 1 (using Normal Approximation)

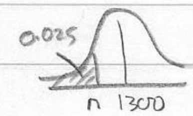
$X \sim B(5000, 0.26)$

$X \approx N(5000 \times 0.26, 5000 \times 0.26 \times (1-0.26))$

$X \approx N(1300, 31.016^2)$

If $P(X<n) < 0.025$

Using "inverse normal" on calculator,



$n \approx 1239$.

Check with "Binomial CD" on calculator

$P(X<1239) = P(X \leq 1238)$

$= 0.02326\ldots$

$P(X<1240) = P(X \leq 1239)$

$= 0.02511\ldots$

Ans: $n = 1239$

Method 2 (using Trial & Improvement)

Use "Binomial CD" only

x	$P(X \leq x)$	Too large / too small
2500	≈ 1	TL
1000	1×10^{-23}	TS
2000	≈ 1	TL
1500	≈ 0.9999	TL
1200	0.000612	TS
1300	0.5074	TL
1250	0.0547	TL
1230	0.0121...	TS
1240	0.0270...	TL
1238	0.02326...	TS
1239	0.02511...	TL

So $P(X \leq 1238) = 0.02326\ldots$

$\therefore$ Ans $n = 1239$