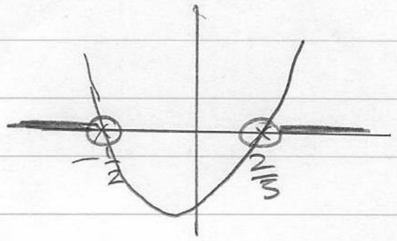
1) $10x^2 + x - 2 > 0$

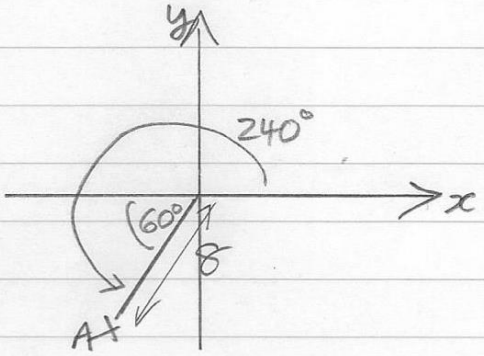
$(5x-2)(2x+1) > 0$

CV. $x = \frac{2}{5}, -\frac{1}{2}$



$x < -\frac{1}{2}, \; x > \frac{2}{5}$ //

2) a) i)

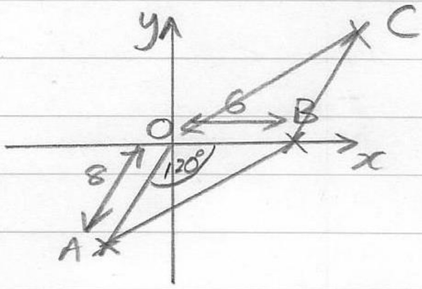


(ii) $\vec{OA} = -(8\cos 60)\underline{i} - (8\sin 60)\underline{j}$

$= -4\underline{i} - 4\sqrt{3}\underline{j}$

b) Area $\triangle AOB = \frac{1}{2}(8)(6)\sin 120$

$= 12\sqrt{3}$ //



c) From sketch $\vec{OA} = \vec{CB}$

$\underline{a} = \underline{b} - \underline{c}$

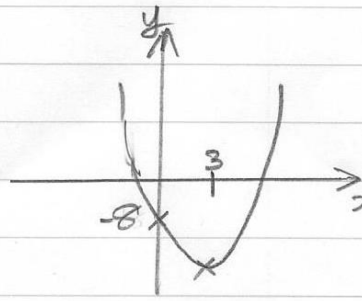
$\underline{c} = \underline{b} - \underline{a} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} - \begin{pmatrix} -4 \\ -4\sqrt{3} \end{pmatrix} = \begin{pmatrix} 10 \\ 4\sqrt{3} \end{pmatrix}$

$= 10\underline{i} + 4\sqrt{3}\underline{j}$ //

3a) $f(x) = (x-3)^2 - 17$ for $x \geq k$.

y-intercept $= (-3)^2 - 17 = -8$

vertex $= (3, -17)$



Inverse only if $f(x)$ is one-to-one

$\therefore$ Least value of k is 3 //

b) $ff(5) = f(f(5))$

$= f((5-3)^2 - 17) = f(-13)$

$= (-13-3)^2 - 17 = 239.$

c) $(x-3)^2-17=x$

$x^2-6x-9-17-x=0$

$x^2-7x-26=0$

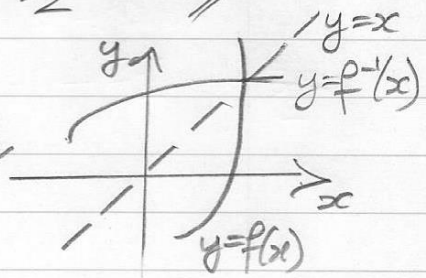
$\cancel{(x\quad)(x\quad)}$ $\quad x=\dfrac{7\pm\sqrt{49+4(26)}}{2}$

$x=\dfrac{7\pm3\sqrt{17}}{2}=9.68\ldots \text{ or } -2.68\ldots$

But domain is $x\geqslant 3$ so $x=\dfrac{7+3\sqrt{17}}{2}$ only //

d) $f^{-1}(x)$ is reflection in line $y=x$

From sketch, all pass through the same point //



4a) Sequence $=\underbrace{16000,\ 17200,\ \ldots}_{\text{AP}\quad a=16000,\ d=1200}$

$u_n=a+(n-1)d$

$\therefore u_{10}=a+9d=16000+9(1200)=£26,800$ //

b) $S_n=\frac{1}{2}n(2a+(n-1)d)$

$=\frac{1}{2}n(32000+(n-1)(1200))$

$=\frac{1}{2}n(1200n+30800)$

$=600n^2+15400n$

If $S_n>500000$ then $600n^2+15400n>500000$

$(\div 2)\quad 3n^2+77n-2500>0$

CV, $n=\dfrac{-77\pm\sqrt{77^2+12(2500)}}{6}$

$n=18.75\ldots \text{ or } -44.42\ldots$

$\therefore n>18.75\ldots$ or $n<-44.42\ldots$

Ans $n=19$ complete years //

c) The model predicts Sam's salary will increase indefinitely.
This is unrealistic //

5a) $x^3 - 3x^2 y + y^2 + 1 = 0$

$$\underbrace{3x^2 - 3x^2\frac{dy}{dx}} + y(-6x) + \underbrace{2y\frac{dy}{dx}} = 0$$

$$\frac{dy}{dx}(2y - 3x^2) = 6xy - 3x^2$$

$$\frac{dy}{dx} = \frac{6xy - 3x^2}{2y - 3x^2}$$

b) At $(1,2)$ $\quad \frac{dy}{dx} = \frac{12-3}{4-3} = 9$

Normal thru $(1,2)$ grad $= -\frac{1}{9}$

$$y - 2 = -\frac{1}{9}(x-1)$$

$$9y - 18 = -x + 1 \quad \Rightarrow \quad x + 9y - 19 = 0 \quad \left(\text{or } y = -\tfrac{1}{9}x + \tfrac{19}{9}\right)$$

6) $f(x) = 2x^3 + 3x$

$$f(x+h) = 2(x+h)^3 + 3(x+h)$$
$$= 2(x^3 + 3x^2h + 3xh^2 + h^3) + 3x + 3h$$
$$= 2x^3 + 6x^2h + 6xh^2 + 2h^3 + 3x + 3h$$

$$f'(x) = \lim_{h\to 0}\left\{\frac{\cancel{2x^3} + 6x^2h + 6xh^2 + 2h^3 + \cancel{3x} + 3h - (\cancel{2x^3} + \cancel{3x})}{h}\right\}$$

$$= \lim_{h\to 0}\left\{6x^2 + 6xh + 2h^2 + 3\right\}$$

$$= 6x^2 + 3$$

7) If $u_n = 25 \times 0.6^n$

$$\text{sequence} = \underbrace{15, 9, 5.4, 3.24, 1.944, \ldots}_{\text{GP} \quad a=15, \quad r=0.6}$$

$$\sum_{n=1}^{\infty} u_n = \underbrace{15 + 9 + 5.4 + \cdots}_{\text{GP} \; a=15, \, r=0.6, \, n=\infty} = S_\infty = \frac{a}{1-r} = \frac{15}{1-0.6} = 37.5$$

$$\sum_{n=1}^{N} u_n = \underbrace{15 + 9 + 5.4 + \cdots}_{\text{GP} \; a=15, \, r=0.6, \, n=N} = S_N = \frac{a(1-r^n)}{1-r} = \frac{15(1-0.6^N)}{1-0.6}$$

$$= 37.5(1 - 0.6^N)$$

If $S_\infty - S_N < 10^{-4}$

then $37.5 - 37.5(1 - 0.6^N) < 10^{-4}$

$$\cancel{37.5} - \cancel{37.5} + 37.5(0.6)^N < 10^{-4}$$

$$(0.6)^N < \frac{1}{375000}$$

$$N \log(0.6) < \log\left(\frac{1}{375000}\right)$$

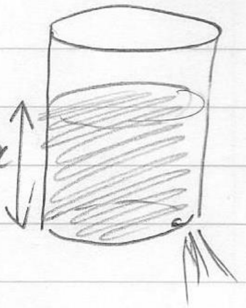
$$N > \frac{\log\left(\frac{1}{375000}\right)}{\log(0.6)}$$

Note $\log(0.6) < 1$ so inequality flips

$$N > 25.125\ldots$$

Smallest value of $N = 26$

8a) $\dfrac{dx}{dt} = -k\sqrt{x}$



If $t=100$, $x=0.64$, $\dfrac{dx}{dt} = -0.0032$

so $\cancel{-}0.0032 = \cancel{-}k\sqrt{0.64}$

$$k = \frac{1}{250} = 0.004$$

$$\therefore \quad \frac{dx}{dt} = -0.004\sqrt{x}$$

b) Separating variables, $\displaystyle\int \frac{dx}{\sqrt{x}} = \int -0.004\, dt$

$$\int x^{-\frac{1}{2}} dx = \int -0.004\, dt$$

GS $\quad 2x^{\frac{1}{2}} = -0.004t + c$

If $x=0.64$, $t=100$ so $\quad 2(0.64)^{\frac{1}{2}} = -0.004(100) + c$

$$1.6 = -0.4 + c \Rightarrow c = 2$$

PS $\quad 2x^{\frac{1}{2}} = 2 - 0.004t$

$$x^{\frac{1}{2}} = 1 - 0.002t$$

PS $\quad x = (1 - 0.002t)^2$

c) If $x=0$, $\quad 0 = (1 - 0.002t)^2$

$$0.002t = 1$$

$$t = 500 \text{ seconds}$$

9a) $R\cos(3x-\alpha) = 3\cos 3x + 7\sin 3x$

$R\cos 3x\cos\alpha + R\sin 3x\sin\alpha = 3\cos 3x + 7\sin 3x$

$\therefore \left.\begin{array}{ll} R\cos\alpha = 3 & ① \\ R\sin\alpha = 7 & ② \end{array}\right\}$ $R=\sqrt{3^2+7^2}$, $R=\sqrt{58}$

② ÷ ① $\frac{\cancel{R}\sin\alpha}{\cancel{R}\cos\alpha} = \frac{7}{3}$

$\tan\alpha = \frac{7}{3} \Rightarrow \alpha = 1.166$ (4sf)

Ans $= \sqrt{58}\cos(3x-1.166)$

b) $\cos x \rightarrow \cos(x-1.166)$ H. translation $+1.166$

$\rightarrow \cos(3x-1.166)$ H. stretch $\times\frac{1}{3}$

$\rightarrow \sqrt{58}\cos(3x-1.166)$ V. stretch $\times\sqrt{58}$

Note: H. stretch must be <u>AFTER</u> the translation
stretch first would give $\cos(3(x-1.166))$

c) ~~R~~ $y = 3\cos 3x + 7\sin 3x$

$y = \sqrt{58}\cos(3x-1.166)$

$y_{max} = \sqrt{58}$ when $3x-1.166 = 0$ or 2π

$3x = 1.166$ or $(2\pi+1.166)$

$\therefore$ smallest positive $x = 0.389$ (3 sf)

d) $y_{min} = -\sqrt{58}$ when $3x-1.166 = -\pi$ or π

$3x = -1.9756...$ or $4.307...$

$\therefore$ smallest positive $x = 1.44$ (3 sf)

10a) Sector $= \frac{1}{2}r^2\theta = \frac{1}{2}(6)^2\theta = 18\theta$

Triangle $= \frac{1}{2}ab\sin C = \frac{1}{2}(6)(6)\sin\theta = 18\sin\theta$

$\therefore$ Area of segment $= 18\theta - 18\sin\theta = 7.2$

(÷18) $\theta - \sin\theta = 0.4$

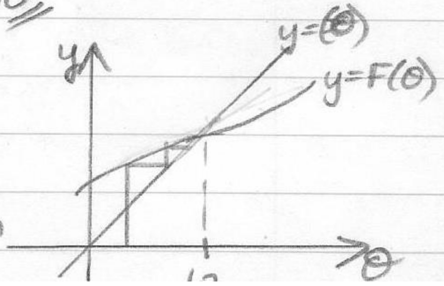
$\theta = 0.4 + \sin\theta$

b) If $F(\theta) = 0.4 + \sin\theta$

$F'(\theta) = \cos\theta$

when $\theta = 1.2$, $F' = \cos 1.2 = 0.3623...$

grad of F is positive and less than 1 $\therefore$ staircase diagram



$\therefore$ Iteration method will converge (assuming 1.2 is close to the root)

c) $\boxed{\theta_{n+1} = 0.4 + \sin\theta_n}$

$\theta_0 = 1.2$

$\theta_1 = 0.4 + \sin 1.2 = 1.332039\ldots$

$\theta_2 = 1.37163\ldots$

$\theta_3 = 1.38023\ldots$

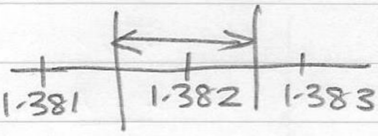
$\theta_4 = 1.38189\ldots$

$\theta_5 = 1.38221\ldots$

$\theta_6 = 1.38227\ldots$ converges to 1.382 (4sf)

d) If $\theta = 0.4 + \sin\theta$

then $G(\theta) = 0.4 + \sin\theta - \theta = 0$



$G(1.3815) = +0.0006368\ldots$
$G(1.3825) = -0.0001754\ldots$ } change of sign & continuous function

$\therefore$ root $= 1.382$ (4sf)

11a) $\int \ln(x-4) \cdot 1 \, dx$

$u = \ln(x-4)$ $\quad \frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{1}{x-4}$ $\quad v = x$

$= x\ln(x-4) - \int \frac{1}{x-4} . x \, dx$

$= x\ln(x-4) - \int \frac{x}{x-4} \, dx$

$$x-4 \overline{)\, x+0} \quad 1 \text{ R}4$$
$$x-4$$
$$4$$

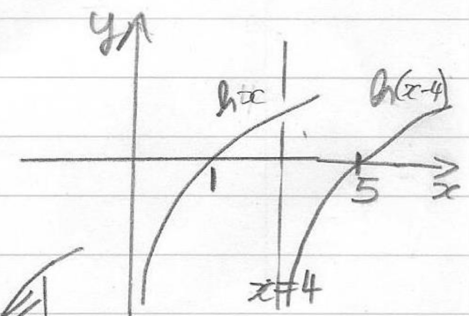
$= x\ln|x-4| - \left[\int \left(1 + \frac{4}{x-4}\right) dx\right]$

$= x\ln|x-4| - \left[x + 4\ln|x-4|\right] + C$

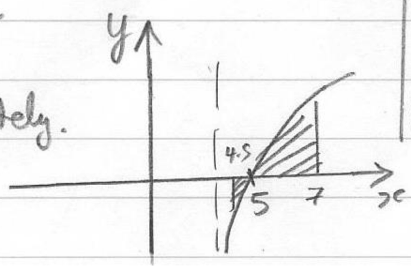
$= (x-4)\ln|x-4| - x + C$

b) $\ln x \rightarrow \ln(x-4)$ H. translation $+4$

V. asymptote at $x = 4$



c) consider regions $4.5 \rightarrow 5$ and $5 \rightarrow 7$ separately.



$$\int_{4.5}^{5} \ln(x-4)\,dx = \Big[(x-4)\ln|x-4| - x\Big]_{4.5}^{5}$$

$$= (1\cancel{\ln 1} - 5) - (0.5\ln 0.5 - 4.5)$$

$$= -5 - 0.5\ln 0.5 + 4.5$$

$$= -0.5 - 0.5\ln 0.5 \quad (= -0.1534\ldots)$$

$$\int_{5}^{7} \ln(x-4)\,dx = \Big[(x-4)\ln|x-4| - x\Big]_{5}^{7}$$

$$= (3\ln 3 - 7) - (1\cancel{\ln 1} - 5)$$

$$= 3\ln 3 - 2 \quad (= +1.2958\ldots)$$

Be careful! Curve below axis, so integral is negative, but area is positive!

$$\text{Total area} = (0.5 + 0.5\ln 0.5) + (3\ln 3 - 2)$$

$$= 0.5\ln\left(\tfrac{1}{2}\right) + 3\ln 3 - 1.5$$

$$= 0.5(\cancel{\ln 1} - \ln 2) + 3\ln 3 - 1.5$$

$$= 3\ln 3 - 0.5\ln 2 - 1.5$$

$$(a = 3,\ b = -0.5,\ c = 1.5)$$

12a)

$$y = a^{3x^2}$$

$$\ln y = \ln\left(a^{3x^2}\right)$$

$$\ln y = 3x^2(\ln a)$$

$$\frac{1}{y}\frac{dy}{dx} = 6x(\ln a)$$

$$\frac{dy}{dx} = 6xy\cdot\ln a = 6xa^{3x^2}\ln a$$

b) At $(1, a^3)$ $\quad \dfrac{dy}{dx} = 6(1)a^3\ln a = 6a^3\ln a$

Tangent thru $(1, a^3)$ grad $= 6a^3\ln a$

$$y - a^3 = 6a^3\ln a\,(x-1)$$

$$y = (6a^3\ln a)x - 6a^3\ln a + a^3$$

when $x = \frac{1}{2}$, $y = 0$ so

$$0 = 6a^3(\ln a)\tfrac{1}{2} - 6a^3\ln a + a^3$$

$$0 = 3a^3\ln a - 6a^3\ln a + a^3$$

$$-3a^3\ln a + a^3 = 0$$

$$a^3(1 - 3\ln a) = 0$$

$$a = 0 \quad \text{or} \quad 1 = 3\ln a$$

$$\ln a = \frac{1}{3} \Rightarrow a = e^{\frac{1}{3}}$$

9) If $\dfrac{dy}{dx} = 6x \cdot a^{3x^2} \ln a = (6\ln a)x \cdot a^{3x^2}$

then $\dfrac{d^2y}{dx^2} = (6\ln a)x \cdot 6x \cdot a^{3x^2} \ln a + a^{3x^2} \cdot (6\ln a)$ using product rule

$$= 36(\ln a)^2 x^2 \cdot a^{3x^2} + (6\ln a) \cdot a^{3x^2}$$

$$= (6\ln a)(a^{3x^2})\left[(6\ln a)x^2 + 1\right] \quad (*)$$

As $a > 1$ (given in question) then $6\ln a > 0$

Also $a^{3x^2} > 0$ for all values of x

And $(6\ln a)x^2 + 1 > 0$ for all x

Therefore $(*)$ is positive for all x

so $\dfrac{d^2y}{dx^2}$ is positive for all x

so curve is convex for all x