Applications of Differential Equations

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The Jacaranda text 'Maths Quest 12 Specialist Mathematics Units 3 & 4 for Queensland' includes Chapter 9 'Applications of Differential Equations'. One question in Exercise 9.5 refers to a chemical reaction with the following information:

Initial volume = 200 litres
Initial salt concentration = 0.1 kg/litre
Rate of outflow = 3 litres per minute
Rate of inflow of salt with concentration 1.5 kg / litre = 2 litres per minute
Contents of container kept stirred

The question asks to set up the DE for the quantity of salt in kg in the container as a function of time, $Q(t)$.

The rate of inflow of salt = 1.5 kg / litre × 2 litres per minute = 3 kg / minute

To calculate the rate of outflow of salt we need to have an expression for the concentration of salt per litre which is constantly changing. Every minute 3 litres of solution is added and 2 litres removed ie a net reduction of 1 litre per minute. Hence at time t the volume of solution is $(200 - t)$ litres.

If $Q(t)$ is the amount of salt at time t then the concentration of salt is $\frac{Q(t)}{200-t}$ kg / litre and since 3 litres of solution is removed per minute then the amount of salt removed is $3\left(\frac{Q(t)}{200-t}\right)$ kg / minute.

The net rate of change of salt is then expressed as a differential equation:

$$\frac{dQ}{dt} = 3 - 3\left(\frac{Q(t)}{200-t}\right)$$

The queston then asks to show that $Q(t) = \frac{3}{2}(200-t) + C(200-t)^3$ is the solution.

First substitute into the DE:

$$\frac{dQ}{dt} = 3 - 3\left(\frac{\frac{3}{2}(200-t) + C(200-t)^3}{200-t}\right)$$

$$= 3 - \frac{9}{2} - 3\,C(200-t)^2 \quad = \quad -\frac{3}{2} - 3\,C(200-t)^2$$

Now find $Q'(t) = \frac{3}{2}(-1) + 3\times C(200-t)^2 = -\frac{3}{2} - 3\,C\,(200-t)^2$, thus showing that $Q(t)$ as given is correct.

The question doesn't require the DE to be solved, just show that the given expression for $Q(t)$ is the solution. Where does this solution come from?

First re-write the DE as $\frac{dQ}{dt} + \frac{3}{200-t} Q = 3$

There are two ways for progressing from here, using insight or a standard method.

1) Insight

Multiply throughout by (200 - t):

$$(200 - t)\frac{dQ}{dt} + 3Q = 3(200 - t)$$

The LHS suggests the product rule, if we multiply by $(200 - t)^{-4}$

$$(200 - t)^{-3}\frac{dQ}{dt} + 3(200-t)^{-4} Q = 3(200 - t)^{-3}$$

Thus $\frac{d}{dt}\left(\frac{Q}{(200-t)^3}\right) = \frac{3}{(200-t)^3}$

Hence $\frac{Q}{(200-t)^3} = \frac{3}{2}\left(\frac{1}{(200-t)^2}\right) + C$

Finally $Q = \frac{3}{2}(200 - t) + C(200 - t)^3$

2) Using an Integrating Factor

Some DEs of the form $y' + p(x)\, y = g(x)$ can be solved by finding an Integrating Factor $\mu(x)$ and multiplying every term of the DE by this factor. We use $\mu(x) = \exp \int^x p(t)\, dt$

Our DE is $\frac{dQ}{dt} + \frac{3}{200-t} Q = 3$ so the Integrating Factor is

$$\mu(t) = \exp \int^t \frac{3}{200-w}\, dw = \exp(-3 \text{ Ln}[\ 200 - t\ ]) = (200 - t)^{-3}$$

Multipling throughout gives

$$(200 - t)^{-3}\frac{dQ}{dt} + 3(200 - t)^{-4} Q = 3(200 - t)^{-3}$$

$$\frac{d}{dt}\left[(200 - t)^{-3} Q\right] = 3(200 - t)^{-3}$$

$$(200 - t)^{-3} Q = \frac{3}{2}(200 - t)^{-2} + C$$

Finally $Q = \frac{3}{2}(200 - t) + C(200 - t)^3$