Using the inverse function to evaluate an integral numerically

Dr Richard Kenderdine
Kenderdine Maths Tutoring

The problem: Find the length of the arc of the function $y = \sqrt{x}$ between $x = 0$ and $x = 4$

The arc length of the function $y = f(x)$ between x = a and $x = b$ is given by

$$\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

For $y = \sqrt{x}$ we have $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$ and hence $\left(\frac{dy}{dx}\right)^2 = \frac{1}{4x}$, thus giving the arc length in this case as

$$\int_0^4 \sqrt{1 + \frac{1}{4x}}\, dx$$

If we try to use numerical integration we find that the function $\sqrt{1 + \frac{1}{4x}}$ is undefined at $x = 0$.

The inverse function is $y = x^2$ and thus $\left(\frac{dy}{dx}\right)^2 = 4x^2$, with new limits $x = 0$ and $x = 2$. The integral is

$$\int_0^2 \sqrt{1 + 4x^2}\, dx$$

This is a standard integral (the primitive of $\sqrt{1 + x^2}$ is $\frac{1}{2}\left(x\sqrt{1 + x^2} + \mathrm{Ln}\left[x + \sqrt{1 + x^2}\right]\right)$) and thus can be easily evaluated.

However, suppose we didn't inow this result and used numerical integration, specifically Simpson's Rule with 5 function values:

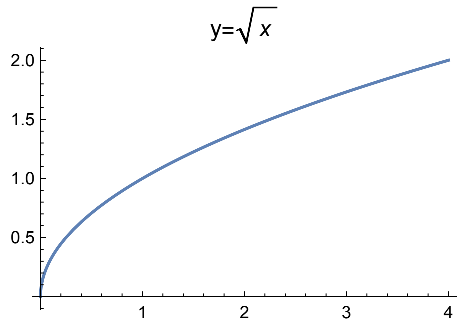
x	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
$y = \sqrt{1 + 4x^2}$	1	$\sqrt{2}$	$\sqrt{5}$	$\sqrt{10}$	$\sqrt{17}$
w	1	4	2	4	1
wy	1	$4\sqrt{2}$	$2\sqrt{5}$	$4\sqrt{10}$	$\sqrt{17}$

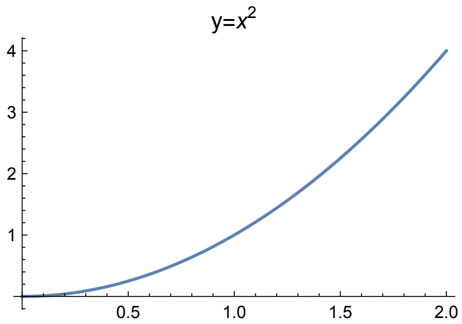
Then $\int_0^2 \sqrt{1 + 4x^2}\, dx = \frac{\frac{1}{2}}{3} \sum w\, y = 4.6468$

Evaluation of the integral with *Mathematica* yields the same result:

$$\int_0^2 \sqrt{1 + 4x^2}\, dx \; // \; N = 4.64678$$

Plot of the original and inverse functions:





Alternative approach

Using the change of variable $x = u^2$ (dx = 2u du) yields

$$\int_0^4 \sqrt{1 + \frac{1}{4\,x}}\; dx = \int_0^2 \sqrt{1 + \frac{1}{4\,u^2}}\; 2\,u\, du$$

$$= \int_0^2 \sqrt{4\,u^2 + 1}\; du$$

Then let v = 2u (dv = 2 du)

$$\int_0^2 \sqrt{4\,u^2 + 1}\; du = \frac{1}{2}\int_0^4 \sqrt{1 + v^2}\; du$$

$$= \frac{1}{2}\left[\frac{v}{2}\sqrt{v^2 + 1} + \frac{1}{2}\,\mathrm{Ln}\left[v + \sqrt{v^2 + 1}\right]\right]_0^4$$

$$= \sqrt{17} + \frac{1}{4}\,\mathrm{Ln}\left[4 + \sqrt{17}\right]$$

$$= 4.64678$$