

# Attractors, Bifurcations and Chaos

Dr Richard Kenderdine

Kenderdine Maths Tutoring

2 January 2015

The logistic map models one-dimensional dynamical systems where the value of a variable in a particular state depends upon the value in the immediately previous state.. A simple model for two-dimensional dynamical systems is the Henon map. This article gives a brief overview.

## Logistic Map

The logistic map is a discrete version of the logistic equation

$$\frac{dx}{dt} = x(1-x) \quad (1)$$

which is used to model population growth as well as a wide variety of other applications.

The logistic map is a recurrence equation

$$x_{n+1} = \alpha x_n(1-x_n) = \alpha(x_n - x_n^2) \quad \alpha > 0 \quad (2)$$

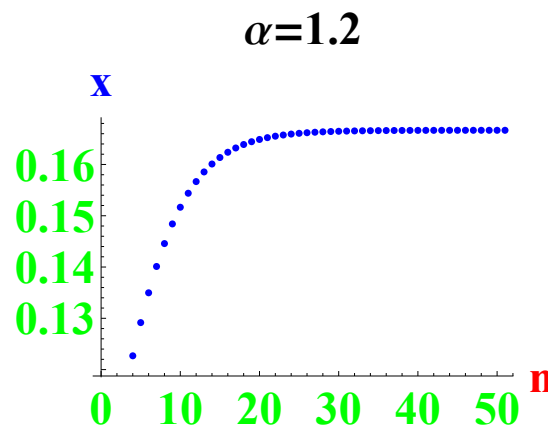
If we consider  $n$  to be a time index then (2) tells us that the value of the system at a given time is a multiple of the difference between the value of the system in the immediately previous time period and the square of that value.

The system starts with initial point  $x_0$  chosen in the interval  $[0, 1]$ . Keeping all subsequent  $x$  values in this interval requires  $\alpha < 4$ , since the maximum value of  $x(1-x)$  is 0.25. The map was developed to model discrete-time population growth with  $x$  being the ratio of existing to maximum population.

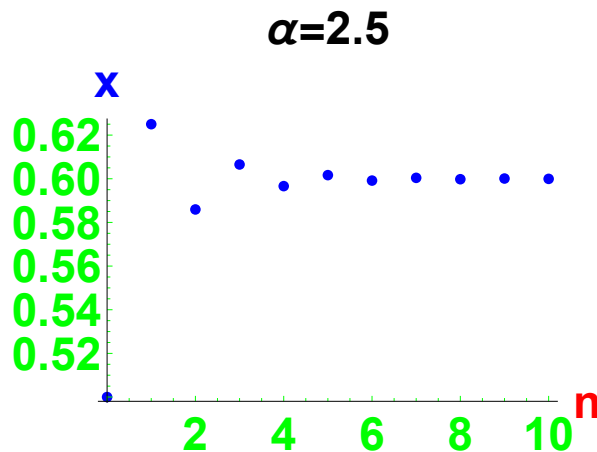
We are interested in determining whether the system converges to a fixed value (a *point of attraction*). If it does then  $x_{n+1} = x_n$  and (2) becomes

$$x_n = \alpha x_n(1-x_n) \quad (3)$$

which has solutions  $x_n = 0, 1 - \frac{1}{\alpha}$ . Obviously the second solution is more interesting. When  $\alpha = 1.2$   $x_n$  converges to  $1 - \frac{1}{1.2} = 0.1\bar{6}$  as shown in the following plot (the value of  $x_0$  is 0.1).

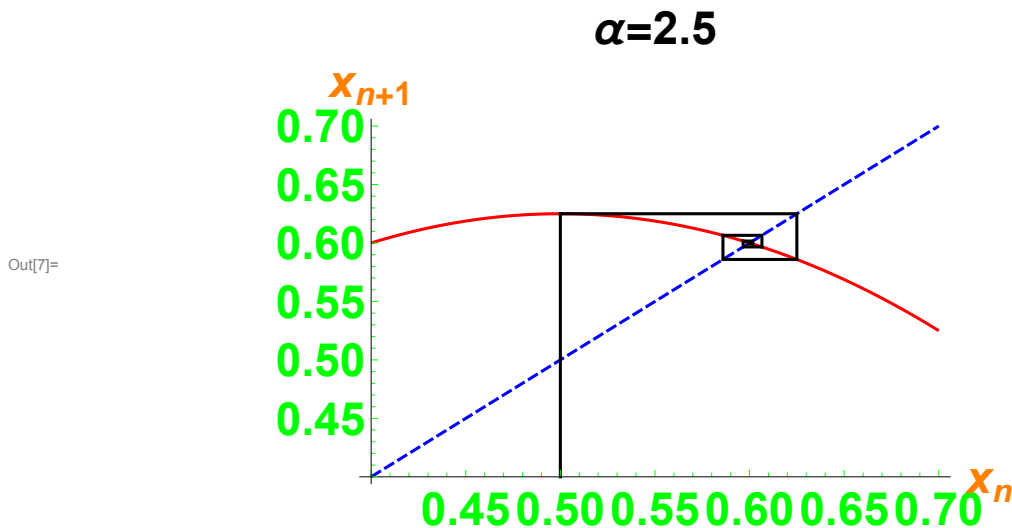


When  $\alpha = 2.5$  the point of attraction is  $x = 1 - \frac{1}{2.5} = 0.6$  and convergence is rapid:

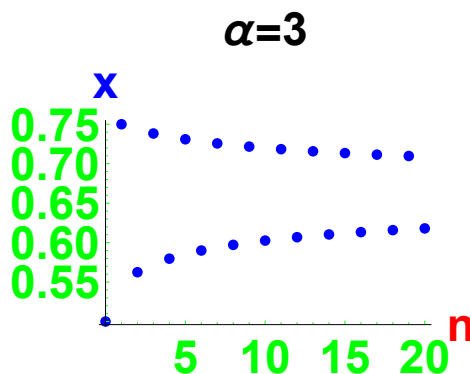


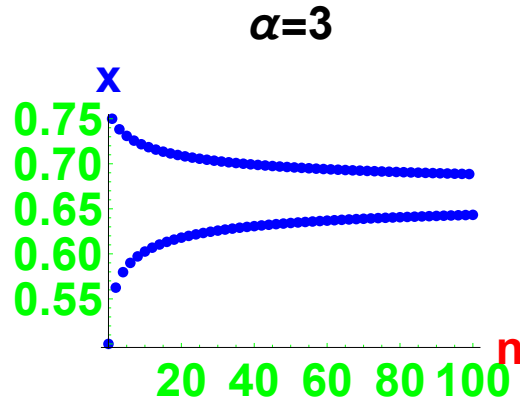
We can use a cobweb diagram to show convergence to a point of attraction. To do this we plot on the same diagram the recurrence function  $x_{n+1} = \alpha x_n (1 - x_n)$  for the five value of  $\alpha$ , together with the line  $x_{n+1} = x_n$  and then show successive values of  $x_n$ . We do this by plotting  $(x_0, 0)$  then draw a line vertically to the recurrence then horizontally to the line. These two steps, vertically to the function then horizontally to the line, are repeated.

When  $\alpha = 2.5$  we see from the above diagram that convergence is rapid. This is clearly shown in the cobweb diagram where the point of attraction is the intersection of the recurrence function and the line  $x_{n+1} = x_n$ :



When  $\alpha = 3$  the system converges very slowly to  $x = \frac{2}{3} = 0.\dot{6}$ . The first plot below shows the first 20 values and the second plot the first 100 values, using  $x_0 = 0.5$ .





However when  $\alpha = 3.1$  the system converges to two alternating points of attraction,  $x = 0.558014$  and  $x = 0.764567$ . The recurrence is said to have period = 2. The points of attraction in this situation are found in the following manner.

We have the recurrence function  $x_{n+1} = \alpha x_n (1 - x_n)$  and also  $x_{n+2} = \alpha x_{n+1} (1 - x_{n+1})$  which can be combined to give

$$x_{n+2} = \alpha[\alpha x_n(1 - x_n)][1 - \alpha x_n(1 - x_n)] \tag{4}$$

and at the fixed point  $x_{n+2} = x_n$ . Thus we have

$$x_n = \alpha[\alpha x_n(1 - x_n)][1 - \alpha x_n(1 - x_n)] \tag{5}$$

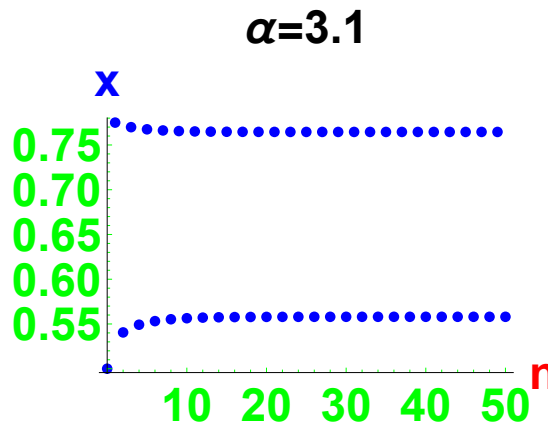
This equation can be written as

$$-\alpha^3 x \left[ x - \left( 1 - \frac{1}{\alpha} \right) \right] \left[ x^2 - \left( 1 + \frac{1}{\alpha} \right) x + \frac{1}{\alpha} \left( 1 + \frac{1}{\alpha} \right) \right] = 0 \tag{6}$$

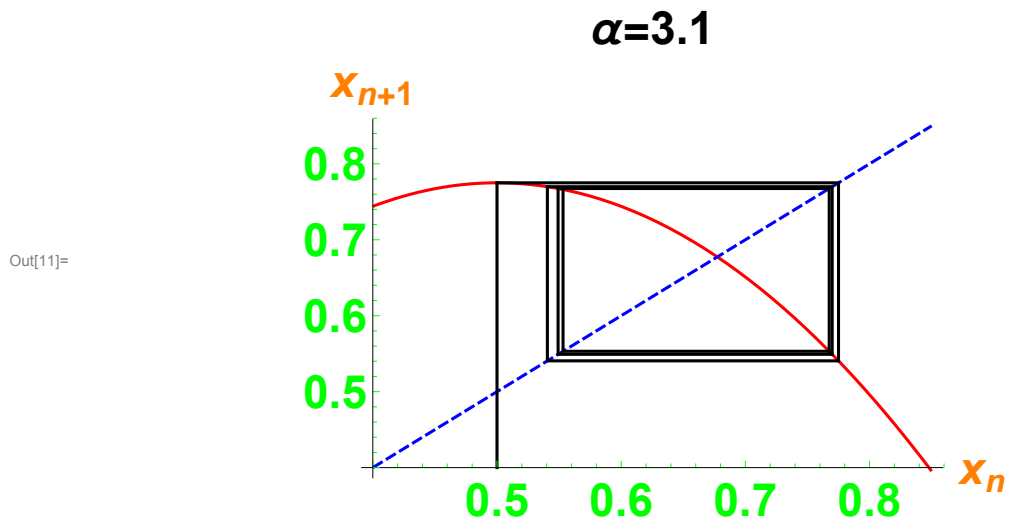
The solutions having period 2 come from the quadratic factor:

$$x = \frac{1}{2} \left[ \left( 1 + \frac{1}{\alpha} \right) \pm \frac{1}{\alpha} \sqrt{(\alpha - 3)(\alpha + 1)} \right] \tag{7}$$

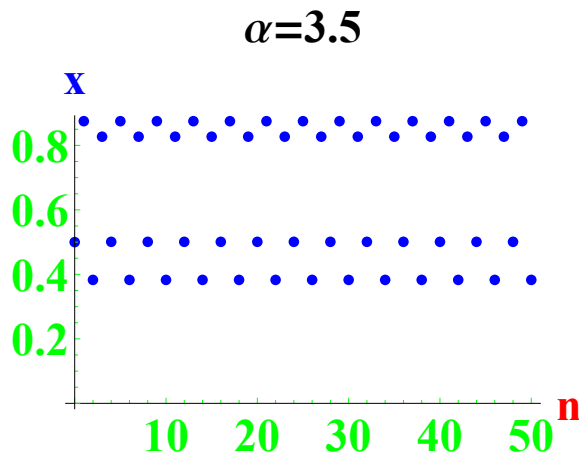
With  $\alpha = 3.1$  we have  $x = 0.558014$  and  $x = 0.764567$  as given above and illustrated in the plot below. Note the rapid convergence.



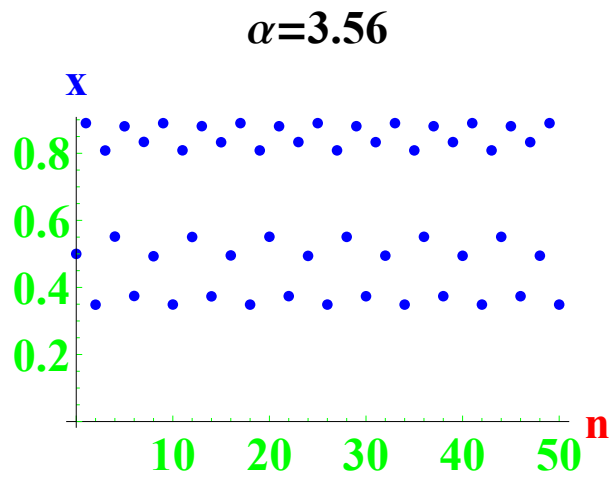
The cobweb diagram shows these points of attraction lying either side of the intersection point of the recurrence function and the line  $x_{n+1} = x_n$ .



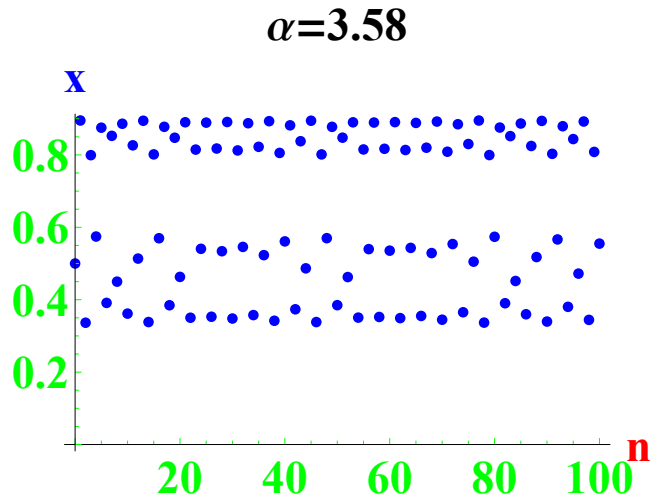
When  $\alpha = 3.5$  there are 4 fixed points, therefore the recurrence has period = 4:



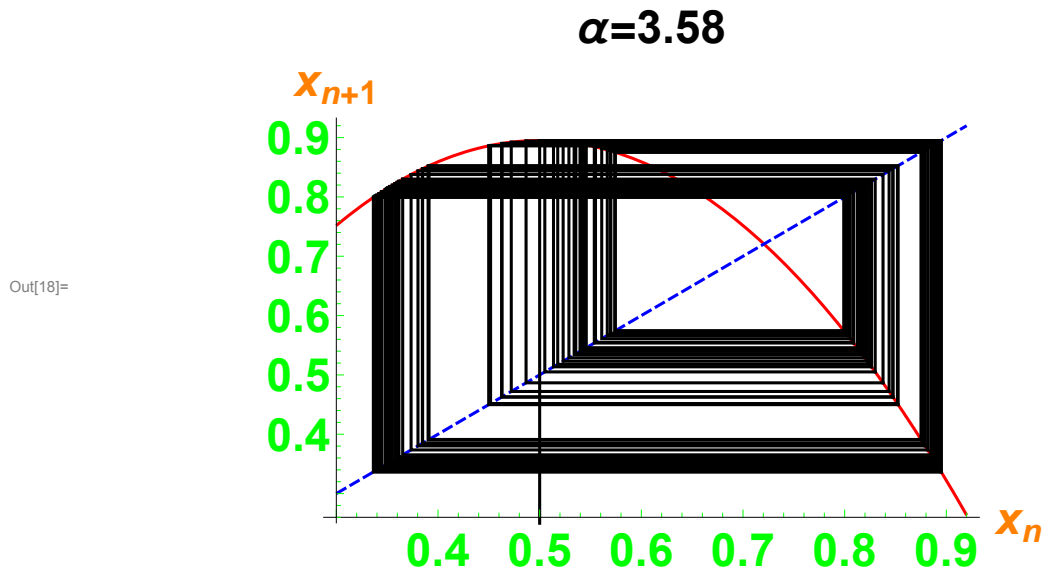
Increasing  $\alpha$  to 3.56 gives 8 fixed points:



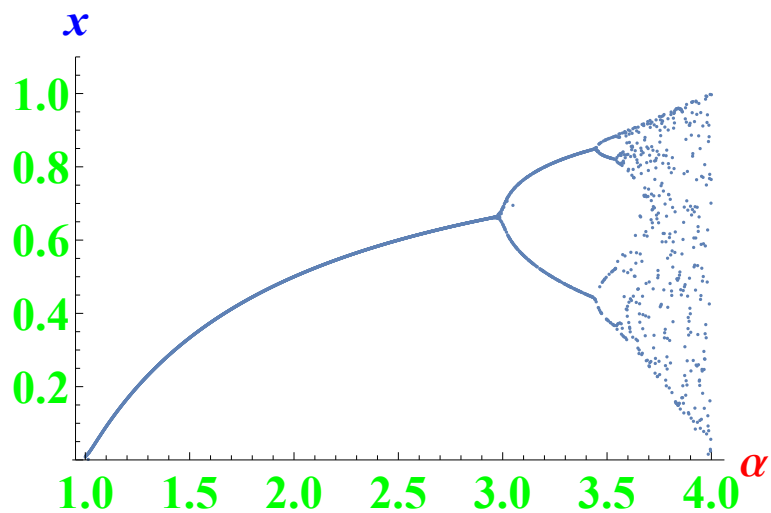
A slight increase to  $\alpha = 3.58$  produces *chaos* in the sense there are no fixed points. The point at which chaos occurs is 3.569945672 and is known as the accumulation point [Ref B].



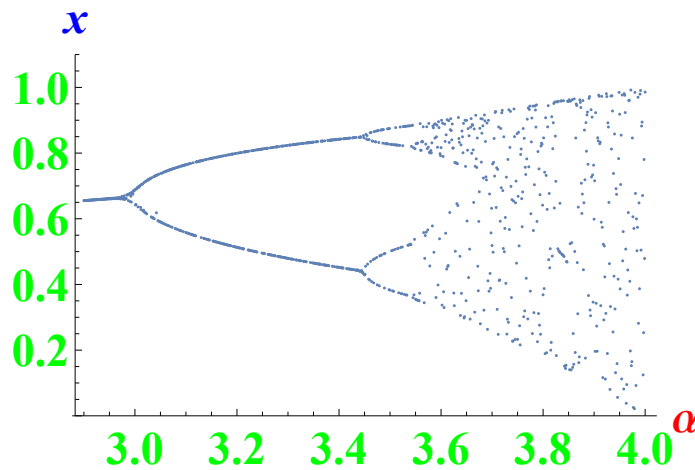
The first 200 values of the recurrence are shown in the cobweb diagram:



We can plot the fixed points against  $\alpha$  in a diagram termed the logistic map:



with an expanded plot for  $2.9 < \alpha < 4$



We see from these maps that there is one fixed point for  $\alpha < 3$ . Then a *pitchfork bifurcation* occurs at  $\alpha = 3$ , giving 2 fixed points. Another bifurcation occurs at  $\alpha = 3.449490$ , giving 4 fixed points. This doubling of the period increases rapidly with 8 fixed points at  $\alpha = 3.544090$  and so on until the accumulation point at  $\alpha = 3.569945672$  [Ref B].

### Henon Map

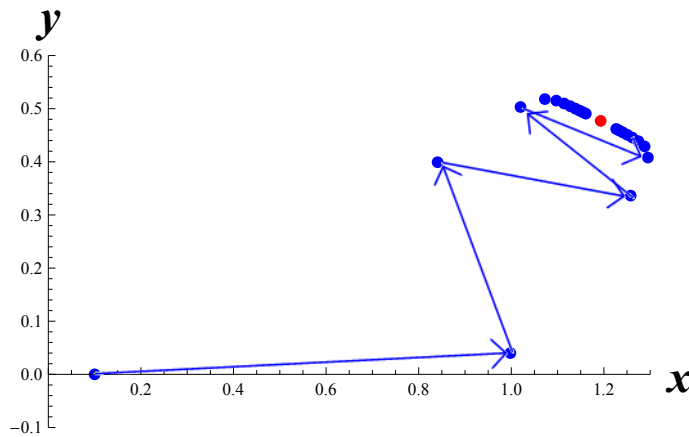
The Henon map is a dynamical system in 2 dimensions, defined by

$$x_{n+1} = 1 + y_n - \alpha x_n^2 \tag{8}$$

$$y_{n+1} = \beta x_n \tag{9}$$

In the following examples  $\beta$  is fixed at 0.4 and the effect of changing  $\alpha$  is illustrated. In the first case  $\alpha = 0.2$  with  $(x_0, y_0) = (0.1, 0)$  and the plot below shows the first 20 points, oscillating about, and converging to, the fixed point of attraction :

$\alpha=0.2$



The arrows indicate the first six points in the series.

The location of the fixed point is determined by solving (8) and (9) simultaneously when  $x_{n+1} = x_n$  and  $y_{n+1} = y_n$ :

$$x_n = 1 + \beta x_n - \alpha x_n^2 \tag{10}$$

Here, with  $\alpha = 0.2$  and  $\beta = 0.4$ , we have the quadratic (dropping the subscript  $n$ )  $x^2 + 3x - 5 = 0$  which has solutions  $x = \frac{-3 \pm \sqrt{29}}{2}$ . The fixed point occurs at  $x = \frac{-3 + \sqrt{29}}{2} = 1.19258$ . (The reason for this is that the absolute value of the eigenvalues of the Jacobian of the system (refer to Appendix) have to **both** be less than 1. The eigenvalues are given by  $\lambda = \alpha x \pm \sqrt{\alpha^2 x^2 + \beta}$  and satisfy the condition when  $x = 1.19258$  [Ref C]).

The  $y$  coordinate is found from  $y = 0.4x = 0.4(1.19258) = 0.47703$ , so the point of attraction is  $(1.19258, 0.47703)$  which is shown in red.

### First and second bifurcations

The first bifurcation occurs when one eigenvalue of the Jacobian matrix equals -1. This results in  $\alpha = 0.75 (1 - \beta)^2$  which in this case gives  $\alpha = 0.75 (1 - 0.4)^2 = 0.27$  [Ref D]

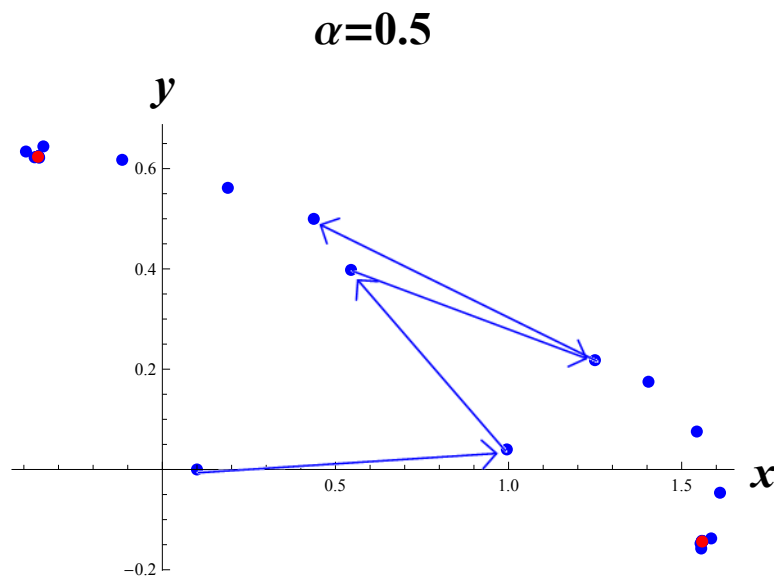
To find the second bifurcation we need the second iteration of the map, that is,  $x_{n+2} = 1 + y_{n+1} - \alpha x_{n+1}^2$  and  $y_{n+2} = \beta x_{n+1}$  with  $y_{n+1} = \beta x_n$  and  $x_{n+1} = 1 + y_n - \alpha x_n^2$ . This gives

$$x_{n+2} = 1 + \beta x_n - \alpha(1 + y_n - \alpha x_n^2)^2 \tag{11}$$

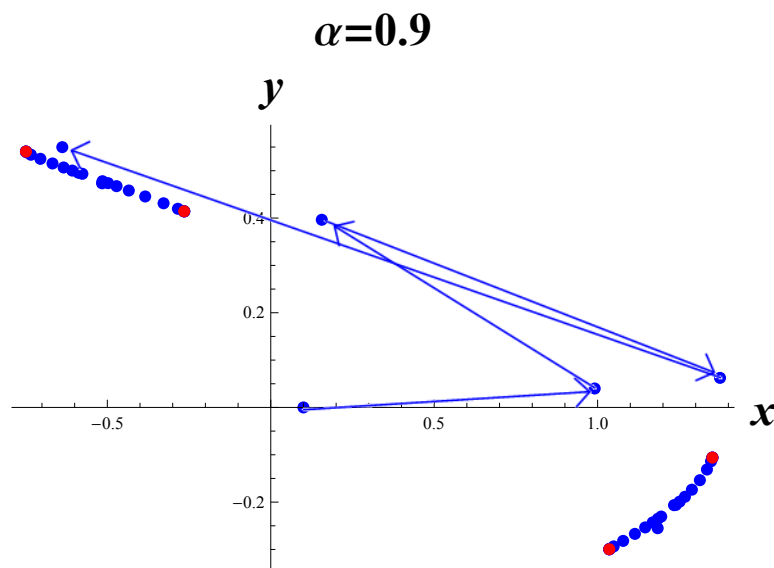
$$y_{n+2} = \beta(1 + y_n - \alpha x_n^2) \tag{12}$$

We then want one eigenvalue of the Jacobian for this iteration to equal -1. The second bifurcation then occurs at  $\alpha = 0.25 (1 + \beta)^2 + (1 - \beta)^2$ ; when  $\beta = 0.4$  this gives  $\alpha = 0.85$ . [Ref D]

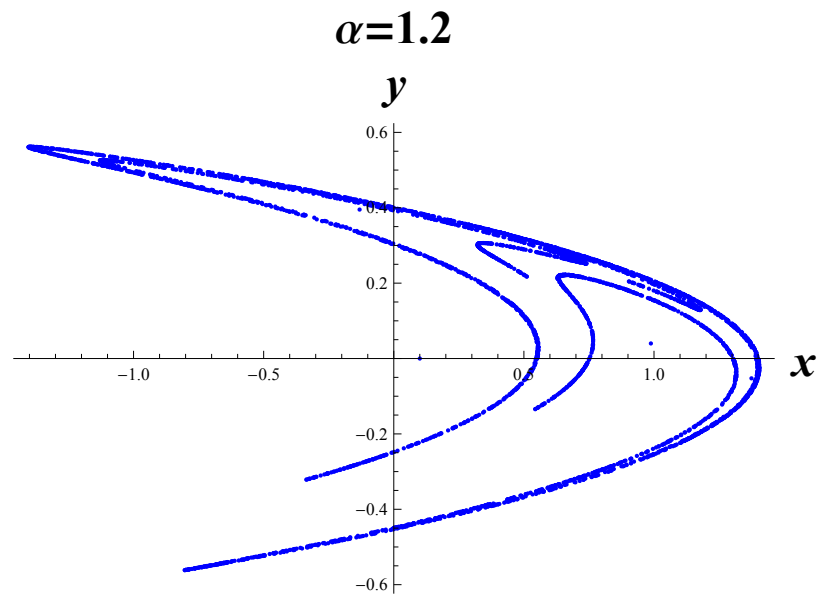
When  $\alpha = 0.5$  there are therefore 2 points of attraction:  $(-0.35917, 0.62367)$  and  $(1.55917, -0.14367)$ , again shown in red in the plot below. These points are found by solving (11) and (12) simultaneously, with  $x_{n+2} = x_n$  and  $y_{n+2} = y_n$ . The resulting equation for  $x$  is  $125x^4 - 500x^2 + 216x + 140 = 0$  which has 4 solutions. The fixed points are again those that result in eigenvalues less than 1 in absolute value [Ref D]. Arrows indicate the first 5 points in the series, subsequent points follow the pattern of oscillating between converging toward the two fixed points.



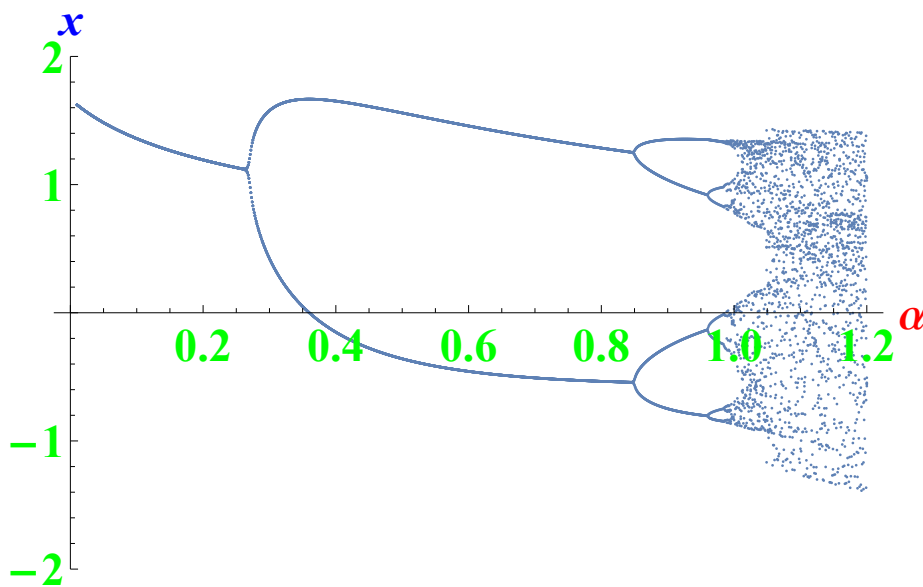
When  $\alpha = 0.9$  there are 4 points of attraction, again shown in red in the following plot together with the first 50 points in the series:



When  $\alpha = 1.2$  there are no fixed points and we have chaos:



We can again plot the fixed points of attraction against  $\alpha$ . The plot below shows that there is one fixed point when  $\alpha < 0.27$ , as given above, and two fixed points for  $0.27 < \alpha < 0.85$ . Increasing  $\alpha$  then leads to chaos.



## Appendix

### The Jacobian and its eigenvalues

The Jacobian is the matrix of partial derivatives of the system equations (8), (9) at the fixed point  $(\bar{x}, \bar{y})$ :

$$\begin{pmatrix} -2\alpha\bar{x} & 1 \\ \beta & 0 \end{pmatrix} \tag{13}$$

We find the eigenvalues,  $\lambda$ , by setting the determinant of

$$\begin{pmatrix} \lambda - 2\alpha\bar{x} & 1 \\ \beta & \lambda \end{pmatrix} \tag{14}$$

equal to 0 ie  $\lambda^2 - 2\alpha\bar{x}\lambda - \beta = 0$ , giving solutions  $\lambda = \alpha x \pm \sqrt{\alpha^2 x^2 + \beta}$



## References

- A. *Logistic Map* <http://mathworld.wolfram.com/LogisticMap.html>
- B. *Logistic Equation*, CRC Concise Encyclopedia of Mathematics, Eric W. Weisstein, Chapman and Hall/CRC, 2003
- C. *Dynamical Properties of the Henon Mapping*, Wadia Faid Hassan Al-Shameri, Int. Journal of Math. Analysis, Vol 6, 2012, no 49, 2419 - 2430 <http://m-hikari.com/ijma/ijma-2012/ijma-49-52-2012/alshameriIJMA49-52-2012.pdf>
- D. *Period Doubling Route to Chaos in a Two Parameter Invertible Map with Constant Jacobian*, Hemants Kr. Sarmah and Ranu Paul, [http://www.arpapress.com/Volumes/Vol3Issue1/IJRRAS\\_3\\_1\\_08.pdf](http://www.arpapress.com/Volumes/Vol3Issue1/IJRRAS_3_1_08.pdf)
- E. *Behavior and attractors in the Henon map*, Kenna Nelson, [http://facultypages.morris.umn.edu/math/Ma4901/F2012/Final/KennaNelson\\_Final.pdf](http://facultypages.morris.umn.edu/math/Ma4901/F2012/Final/KennaNelson_Final.pdf)
- F. *Two Dimensional Henon Map with the Parameter Values  $1 < a < 2, |b| < 1$  in Dynamical Systems*, Md. Jahurul Islam, Md. Shahidul Islam and MA Rahman, <http://www.researchmathsci.org/apamart/apam-v2n2-6.pdf>