Working with bases in *n*-dimensional space

Dr Richard Kenderdine

A basis in *n*-dimensional space is a set of vectors, each with *n* elements, such that each point in the space can be obtained from a linear combination of the vectors.

For example, the standard bases in 2- and 3-dimensional space are {(1, 0), (0, 1)} and {(1,0,0), (0,1,0), (0,0,1)} respectively.

The co-ordinates of a point in terms of one basis can be related to the co-ordinates using another basis through the relationship

$$\text{Matrix of basis vectors using basis } \mathcal{B} \times \text{co-ordinates using basis } \mathcal{B}$$
$$= \text{Matrix of basis vectors using basis } \mathcal{C} \times \text{co-ordinates using basis } \mathcal{C} \qquad \textbf{(1)}$$

For example, suppose we have the point (-6,11) using the standard basis and we want to calculate the co-ordinates using basis {(2,1), (-1,3)}.

Using **(1)** we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -6 \\ 11 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

We can either use simultaneous equations ($2x - y = -6$ and $x + 3y = 11$) to obtain $(x, y) = (-1, 4)$ or use the inverse of $\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$ and then

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\frac{1}{7}\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -6 \\ 11 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

This result can also be expressed as a linear combination of the basis vectors:

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix}(-1) + \begin{pmatrix} -1 \\ 3 \end{pmatrix}(4) = \begin{pmatrix} -6 \\ 11 \end{pmatrix}$$

Using a transition matrix

We can use a transition matrix to change from one non-standard basis to another. Denote $\mathcal{P}$ as the transition matrix from base $\mathcal{B}$ to base $\mathcal{C}$. Then $\mathcal{B}\,\mathcal{P} = \mathcal{C}$

For example, let $\mathcal{B} = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$ and $\mathcal{C} = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$ then

$$\mathcal{P} = \mathcal{B}^{-1}\,\mathcal{C} = \frac{1}{7}\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 2 & 9 \\ -3 & 4 \end{pmatrix}$$

Check:
$$\mathcal{B}\,\mathcal{P} = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}\frac{1}{7}\begin{pmatrix} 2 & 9 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} = \mathcal{C}$$

We also need $\mathcal{P}^{-1} = \frac{1}{5}\begin{pmatrix} 4 & -9 \\ 3 & 2 \end{pmatrix}$ to calculate $\mathcal{B}$ from $\mathcal{C}$:

$$\mathcal{C}\,\mathcal{P}^{-1} = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \frac{1}{5}\begin{pmatrix} 4 & -9 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} = \mathcal{B} \text{ as required}$$

Having obtained the transition matrix we can now calculate the coordinates of a point under another basis using **(1)** . Continuing the above example where we had (-6,11) as coordinates with the standard basis that became (-1, 4) with basis $\mathcal{B}$. Now we want to calculate the coordinates under basis $\mathcal{C}$ ie we need to find coordinates (x, y) such that, using **(1)**

$$\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Introduce $\mathcal{P}\mathcal{P}^{-1}$ on the left

$$\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} \frac{1}{7}\begin{pmatrix} 2 & 9 \\ -3 & 4 \end{pmatrix} \frac{1}{5}\begin{pmatrix} 4 & -9 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

Multiplying the first two matrices yields $\mathcal{C}$ so the coordinates are obtained from

$$\frac{1}{5}\begin{pmatrix} 4 & -9 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} -8 \\ 1 \end{pmatrix}.$$

As a check, multiplying these coordinates by the basis vectors should yield the coordinates using the standard basis:

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}(-8) + \begin{pmatrix} 2 \\ 3 \end{pmatrix}(1) = \begin{pmatrix} -6 \\ 11 \end{pmatrix} \text{ as required.}$$

A 3-dimensional example

Let $\mathcal{B} = \begin{pmatrix} 2 & 2 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$ and $\mathcal{C} = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 1 & 0 \\ -5 & -3 & 2 \end{pmatrix}$ then the long way of calculating the transition matrix from $\mathcal{B}$ to $\mathcal{C}$ is to use simultaneous equations, initially solving

$$\begin{pmatrix} 3 \\ 1 \\ -5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} a + \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} b + \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} c$$

The solution is $a = \frac{35}{2}$, $b = \frac{-19}{2}$ and $c = -13$. This is the first column of the transition matrix that can be obtained more efficiently using $\mathcal{B}\,\mathcal{P} = \mathcal{C}$ and hence $\mathcal{P} = \mathcal{P}^{-1}\,\mathcal{C}$

We have $\mathcal{B}^{-1} = \frac{1}{2}\begin{pmatrix} 3 & 1 & -5 \\ -1 & -1 & 3 \\ -2 & 0 & 4 \end{pmatrix}$ and hence $\mathcal{P} = \mathcal{B}^{-1}\,\mathcal{C} = \frac{1}{2}\begin{pmatrix} 35 & 19 & -13 \\ -19 & -11 & 7 \\ -26 & -14 & 10 \end{pmatrix}$

To convert from $\mathcal{C}$ to $\mathcal{B}$ we use $\mathcal{P}^{-1} = \begin{pmatrix} 3 & 2 & \frac{5}{2} \\ -2 & -3 & \frac{-1}{2} \\ 5 & 1 & 6 \end{pmatrix} = \mathcal{C}^{-1}\,\mathcal{B}$

Suppose we have the point (-5, 8, -5) with the standard basis. The coordinates with basis $\mathcal{B}$ are

$$\mathcal{B}^{-1}\begin{pmatrix} -5 \\ 8 \\ -5 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3 & 1 & -5 \\ -1 & -1 & 3 \\ -2 & 0 & 4 \end{pmatrix}\begin{pmatrix} -5 \\ 8 \\ -5 \end{pmatrix} = \begin{pmatrix} 9 \\ -9 \\ -5 \end{pmatrix}$$

while under basis $\mathcal{C}$ they are

$$\mathcal{C}^{-1}\begin{pmatrix} -5 \\ 8 \\ -5 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ -1 & \frac{1}{2} & \frac{-1}{2} \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} -5 \\ 8 \\ -5 \end{pmatrix} = \begin{pmatrix} \frac{-7}{2} \\ \frac{23}{2} \\ 6 \end{pmatrix}$$

To check the direct connection between these coordinates we use **(1)** and the transition matrix:

$$\begin{pmatrix} 2 & 2 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 9 \\ -9 \\ -5 \end{pmatrix} = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 1 & 0 \\ -5 & -3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 1 \end{pmatrix}\mathcal{P}\mathcal{P}^{-1}\begin{pmatrix} 9 \\ -9 \\ -5 \end{pmatrix} = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 1 & 0 \\ -5 & -3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 & -1 \\ 1 & 1 & 0 \\ -5 & -3 & 2 \end{pmatrix}\mathcal{P}^{-1}\begin{pmatrix} 9 \\ -9 \\ -5 \end{pmatrix} = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 1 & 0 \\ -5 & -3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \text{using } \mathcal{B}\,\mathcal{P} = \mathcal{C}$$

then cancelling $\mathcal{C}$ from both sides we have

$$\begin{pmatrix} 3 & 2 & \frac{5}{2} \\ -2 & -3 & \frac{-1}{2} \\ 5 & 1 & 6 \end{pmatrix}\begin{pmatrix} 9 \\ -9 \\ -5 \end{pmatrix} = \begin{pmatrix} \frac{-7}{2} \\ \frac{23}{2} \\ 6 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ as before}$$