Linear Algebra - Change of Basis

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In Linear Algebra any vector in a vector space can be expressed as a linear combination of basis vectors in that space. For example, in 2D the standard basis is ((1, 0), (0, 1)). For any vector space there is more than one set of basis vectors and it is useful to know how to transition from one basis to another.

Let B_1 and B_2 be two matrices of basis vectors and let $\boldsymbol{u}$ and $\boldsymbol{v}$ be the co-ordinates of a single point relative to the respective bases.

Then $B_1\boldsymbol{u} = B_2\boldsymbol{v}$ ie the product of the matrix of basis vectors and the co-ordinates of the point under that basis is constant over all bases.

Let T be the transition matrix from B_1 to B_2. That is, $B_1T = B_2$ so then $T = B_1^{-1}B_2$ and T^{-1} is the transition matrix from B_2 to B_1.

It follows from $B_1\boldsymbol{u} = B_2\boldsymbol{v}$ that $B_1^{-1}B_1\boldsymbol{u} = B_1^{-1}B_2\boldsymbol{v} \Rightarrow \boldsymbol{u} = B_1^{-1}B_2\boldsymbol{v} = T\boldsymbol{v}$ or, alternatively, $T^{-1}\boldsymbol{u} = \boldsymbol{v}$

It might seem strange that multiplying the transition matrix from B_1 to B_2 by the co-ordinates under B_2 gives the co-ordinates under B_1. Hopefully the derivation of this result makes it clear that this is correct.

Example in 2D

Let B_1= ((2, 3), (1, -1)), $B_2 = ((1, -4), (3, 1))$ and T be the transition matrix from B_1 to B_2.

That is, $\begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix} T = \begin{pmatrix} 1 & 3 \\ -4 & 1 \end{pmatrix}$

Hence $T = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 3 \\ -4 & 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ -4 & 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 11 & 7 \end{pmatrix}$

Then $T^{-1} = \frac{1}{13}\begin{pmatrix} -7 & 4 \\ 11 & 3 \end{pmatrix}$, this is the transition matrix from B_2 to B_1

Suppose we have a point with co-ordinates (3, -2) using B_1, hence the co-ordinates under the standard basis are $\begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 11 \end{pmatrix}$, using constancy under all bases.

Then, under B_2 we have $\begin{pmatrix} 1 & 3 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 11 \end{pmatrix} \Rightarrow \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ -4 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 4 \\ 11 \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 1 & -3 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 11 \end{pmatrix}$

Thus $$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \frac{1}{13}\begin{pmatrix} -29 \\ 27 \end{pmatrix}$$

This result could have been obtained directly using $T^{-1}(x_1, y_1) = (x_2, y_2)$

ie $\frac{1}{13}\begin{pmatrix} -7 & 4 \\ 11 & 3 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \frac{1}{13}\begin{pmatrix} -29 \\ 27 \end{pmatrix}$

Obtaining transition matrices another way

The transition matrix can be obtained by augmenting one set of basis vectors with the other set and row reducing one set to the identity matrix. For example, to obtain the transition matrix from B_1 to B_2 in the above we have

$$\begin{pmatrix} 2 & 1 & 1 & 3 \\ 3 & -1 & -4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0.5 & 0.5 & 1.5 \\ 3 & -1 & -4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0\cdot5 & 0.5 & 1.5 \\ 0 & -2\cdot5 & -5\cdot5 & -3\cdot5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0\cdot5 & 0\cdot5 & 1.5 \\ 0 & 1 & 2\cdot2 & 1.4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -0.6 & 0.8 \\ 1 & 0 & 2.2 & 1.4 \end{pmatrix}$$

Hence the transition matrix from B_1 to B_2 is $\frac{1}{5}\begin{pmatrix} -3 & 4 \\ 11 & 7 \end{pmatrix}$ as previously.

The transition matrix from B_2 to B_1 can be found similarly (instead of inverting $\frac{1}{5}\begin{pmatrix} -3 & 4 \\ 11 & 7 \end{pmatrix}$)

$$\begin{pmatrix} 1 & 3 & 2 & 1 \\ -4 & 1 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 & 1 \\ 0 & 13 & 11 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 & 1 \\ 0 & 1 & \frac{11}{13} & \frac{3}{13} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\frac{7}{13} & \frac{4}{13} \\ 0 & 1 & \frac{11}{13} & \frac{3}{13} \end{pmatrix}$$

Hence the transition matrix from B_2 to B_1 is $\frac{1}{13}\begin{pmatrix} -7 & 4 \\ 11 & 3 \end{pmatrix}$, as previously.

Example: Rotation of axes

As an example of a transition matrix consider rotating the axes by angle α to new axes x', y' as shown in Figure 1.

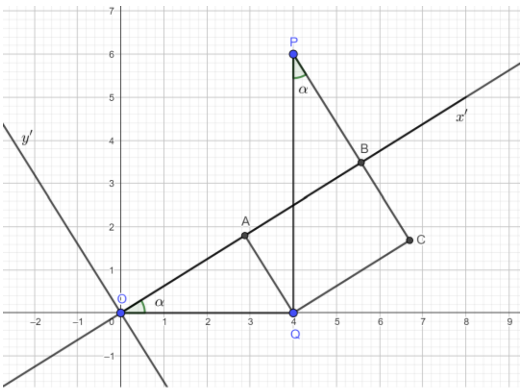

Figure 1: Rotation of axes by angle α

P is the point (4, 6) using the standard axes. For illustrative purposes we can generalise the co-ordinates to (x, y).

Then *OQ* = x, *OA* = $x\cos\alpha$, *AQ* = $x\sin\alpha$, *PQ* = y, *PC* = $y\cos\alpha$ and *QC* = $y\sin\alpha$.

Hence *OB* = *OA* + *AB* = *OA* + *QC* = $x\cos\alpha + y\sin\alpha$ and *PB* = *PC* – *BC* = *PC* – *AQ* = $y\cos\alpha - x\sin\alpha$.

Using the rotated axes $x' = OB$ and $y' = PB$ hence

$$x' = x\cos\alpha + y\sin\alpha$$
$$y' = -x\sin\alpha + y\cos\alpha$$

In matrix form we have

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

And in reverse

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$