A combinatoric question that reveals the 'hockey-stick' identity.

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The 'hockey-stick' identity for Pascal's Triangle equates the sum of certain binomial coefficients to a single binomial coefficient. The question in this note concerns the constrained selection of elements in a set and reveals the 'hockey-stick' identity in the solution. A simpler symbolic method is then used to achieve the same result.

Question

Determine the number of ways of choosing five numbers from the first eighteen positive integers such that any two chosen numbers differ by at least two.

A) Solution by enumeration of all cases

The solution will be obtained by first fixing the smaller numbers and counting possible values for the last number then progressively increasing the smaller numbers.

1) **Possible values for the fifth number with first three numbers fixed.**

The first four numbers are kept as small as possible and all possible values for the fifth number are considered. The chosen values are in red:

1, 2, 3, 4, 5, 6, 7, 8, ,9, 10, 11, 12, 13, 14, 15, 16, 17, 18   10 ways

Now increase the fourth number to 8

1, 2, 3, 4, 5, 6, 7, 8, ,9, 10, 11, 12, 13, 14, 15, 16, 17, 18   9 ways

Continue to increase the fourth number until 16, each time the possible values for the fifth number decreases by 1. Table 1 shows the results.

1st	2nd	3rd	4th	5th	Number of ways
1	3	5	7	9 – 18	10
1	3	5	8	10 – 18	9
1	3	5	9	11 – 18	8
1	3	5	10	12 – 18	7
1	3	5	11	13 – 18	6
1	3	5	12	14 – 18	5
1	3	5	13	15 – 18	4
1	3	5	14	16 – 18	3
1	3	5	15	17, 18	2
1	3	5	16	18	1
					55

Table 1: First three numbers fixed.

2) **Changing the third number**
Now increase the third number to 6 and repeat the process.

1, 2, 3, 4, 5, 6, 7, 8, ,9, 10, 11, 12, 13, 14, 15, 16, 17, 18   9 ways

Progressively increase the fourth number from 8 to 16 and count the number of possible choices for the fifth number, as shown in Table 2.

1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}	Number of ways
1	3	6	8	10 – 18	9
1	3	6	9	11 – 18	8
1	3	6	10	12 – 18	7
1	3	6	11	13 – 18	6
1	3	6	12	14 – 18	5
1	3	6	13	15 – 18	4
1	3	6	14	16 – 18	3
1	3	6	15	17, 18	2
1	3	6	16	18	1
					45

Table 2: Third number increased from 5 to 6

Now increase the third number to 7, as shown in Table 3.

1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}	Number of ways
1	3	7	9	11 – 18	8
1	3	7	10	12 – 18	7
1	3	7	11	13 – 18	6
1	3	7	12	14 – 18	5
1	3	7	13	15 – 18	4
1	3	7	14	16 – 18	3
1	3	7	15	17, 18	2
1	3	7	16	18	1
					36

Table 3: Third number increased to 7.

This is repeated until the third number reaches 14 when we have {1, 3, 14, 16, 18} as the single possible selection. The total number of ways of choosing three numbers to combine with {1, 3} is:

$$55 + 45 + 36 + 28 + 21 + 15 + 10 + 6 + 3 + 2 + 1 = 220$$

3) **Increase the second number to 4 and 5.**

Now repeat the process with the second number being 4.

1, 2, 3, 4, 5, 6, 7, 8, ,9, 10, 11, 12, 13, 14, 15, 16, 17, 18 9 ways

1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}	Number of ways
1	4	6	8	10 – 18	9
1	4	6	9	11 – 18	8
1	4	6	10	12 – 18	7
1	4	6	11	13 – 18	6
1	4	6	12	14 – 18	5
1	4	6	13	15 – 18	4
1	4	6	14	16 – 18	3
1	4	6	15	17, 18	2
1	4	6	16	18	1
					45

Table 4: Second number increased to 4.

The third number will now increase from 6 in steps of 1 to 14 and the total number of ways is

$$45 + 36 + 28 + 21 + 15 + 10 + 6 + 3 + 2 + 1 = 165$$

Increasing the second number to 5 means the third number can start at 7.

1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}	Number of ways
1	5	7	9	11 – 18	8
1	5	7	10	12 – 18	7
1	5	7	11	13 – 18	6
1	5	7	12	14 – 18	5
1	5	7	13	15 – 18	4
1	5	7	14	16 – 18	3
1	5	7	15	17, 18	2
1	5	7	16	18	1
					36

Table 5: Second number increased to 5.

The total number of ways when the second number is 5 and the third number increases from 7 to 14 is

$$36 + 28 + 21 + 15 + 10 + 6 + 3 + 2 + 1 = 120$$

4) **The solution**

The pattern is now emerging and can be summarised in Table 6.

1^{st} number	Number of ways for 5^{th} number	Cumulative 1 Moving 4^{th}	Cumulative 2 Moving 3^{rd}	Cumulative 3 Moving 2^{nd}
1	10	55	220	715
2	9	45	165	495
3	8	36	120	330
4	7	28	84	210
5	6	21	56	126
6	5	15	35	70
7	4	10	20	35
8	3	6	10	15
9	2	3	4	5
10	1	1	1	1
Total	55	220	715	2002

Table 6: Number of possible selections when increasing the numbers in turn.

Therefore, the total number of ways of choosing five numbers from eighteen such that all the numbers are at least two apart is 2002.

Now comes the interesting part.

The sum of the first n integers is given by $\frac{n(n+1)}{2}$ and these sums are called triangular numbers because they look like triangles when the integers being summed are written in symbolic form. For example, 1 + 2 + 3 + 4 + 5 + 6 = 21, shown as a triangle with twenty-one 1s:

1 1
2 1 1
3 1 1 1
4 1 1 1 1
5 1 11 1 1
6 1 1 1 1 1 1

The triangular numbers are the entries in the Column 3, the result of moving the fourth number to all possible positions, given the position of the first number. Note that these entries accumulate the entries in Column 2.

Looking at the first two columns from a combinatoric view point we note that Column 2 is $\binom{n}{1}$ for n from 10 to 1 (reading down the column) while Column 3 is $\binom{n+1}{2}$, as this is the same as the formula for the triangular numbers, $\frac{n(n+1)}{2}$.

Notice that Column 4, obtained by moving the third number to all possible positions, accumulates Column 3, hence finding the sum of the triangular numbers. The required formula is

$$\sum_{k=1}^{n} \frac{k(k+1)}{2} = \frac{1}{6} n(n+1)(n+2)$$

When n = 10 we have $\frac{1}{6}(10)(11)(12) = 220$. Note that $\frac{1}{6}n(n+1)(n+2) = \binom{n+2}{3}$.

Column 5 results from moving the second number to all possible positions and accumulates Column 4. The required formula is

$$\sum_{k=1}^{n} \frac{1}{6} k(k+1)(k+2) = \frac{1}{24} n(n+1)(n+2)(n+3)$$

When n = 10 we have $\frac{1}{24}(10)(11)(12)(13) = 715$. Note that $\frac{1}{24}n(n+1)(n+2)(n+3) = \binom{n+3}{4}$.

There is an obvious pattern here. The sum of the fifth column gives the total number of ways of choosing the five numbers for all possible positions of the first number and should therefore be, following the pattern, $\binom{n+4}{5}$ for $n = 10$ ie $\binom{14}{5}$ which is 2002.

This analysis shows the hockey-stick property of Pascal's Triangle in operation. Each of the Columns 2 to 5 satisfy:

$$\sum_{k=r}^{n} \binom{k}{r} = \binom{n+1}{r+1}$$

For example, Column 3 satisfies:

$$\sum_{k=1}^{10} \binom{k+1}{2} = \binom{10+1+1}{3} = \binom{12}{3} = 220$$

That is,

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} + \binom{6}{2} + \binom{7}{2} + \binom{8}{2} + \binom{9}{2} + \binom{10}{2} + \binom{11}{2} = \binom{12}{3}$$

```
{1}
{1, 1}
{1, 2, 1}
{1, 3, 3, 1}
{1, 4, 6, 4, 1}
{1, 5, 10, 10, 5, 1}
{1, 6, 15, 20, 15, 6, 1}
{1, 7, 21, 35, 35, 21, 7, 1}
{1, 8, 28, 56, 70, 56, 28, 8, 1}
{1, 9, 36, 84, 126, 126, 84, 36, 9, 1}
{1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1}
{1, 11, 55, 165, 330, 462, 462, 330, 165, 55, 11, 1}
{1, 12, 66, 220, 495, 792, 924, 792, 495, 220, 66, 12, 1}
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As each Column 3 to 5 accumulates the entries in the previous column then each entry in these columns also satisfy the hockey-stick property. For example, the second entry in Column 5 satisfies:

$$\sum_{k=1}^{9} \binom{k+2}{3} = \binom{9+2+1}{4} = \binom{12}{4} = 495$$

Checking with *Mathematica*

The summations are checked first:

In[1]:= $\sum_{k=1}^{n} \frac{k\ (k+1)}{2}$

Out[1]= $\frac{1}{6}\, n\ (1+n)\ (2+n)$

In[3]:= $\sum_{k=1}^{n} \left(\frac{1}{6}\, k\ (k+1)\ (k+2)\right)$

Out[3]= $\frac{1}{24}\, n\ (1+n)\ (2+n)\ (3+n)$

In[5]:= $\sum_{k=1}^{n} \frac{1}{24}\, k\ (1+k)\ (2+k)\ (3+k)$

Out[5]= $\frac{1}{120}\, n\ (1+n)\ (2+n)\ (3+n)\ (4+n)$

Now obtain the values for the accumulation of entries in Column 5:

In[6]:= Table$\left[\frac{1}{120}\, n\ (1+n)\ (2+n)\ (3+n)\ (4+n),\ \{n, 1, 10\}\right]$

Out[6]= {1, 6, 21, 56, 126, 252, 462, 792, 1287, 2002}

Then obtain all possible selections of five numbers and check that the total is 2002 (the first ten and last nine are shown):

```
In[14]:= Flatten[Table[{a, b, c, d, e}, {a, 1, 10},
           {b, a+2, 12}, {c, b+2, 14}, {d, c+2, 16}, {e, d+2, 18}], 4]

Out[14]= {{1, 3, 5, 7, 9}, {1, 3, 5, 7, 10}, {1, 3, 5, 7, 11},
          {1, 3, 5, 7, 12}, {1, 3, 5, 7, 13}, {1, 3, 5, 7, 14}, {1, 3, 5, 7, 15},
          {1, 3, 5, 7, 16}, {1, 3, 5, 7, 17}, {1, 3, 5, 7, 18}, ... 1983 ...,
          {8, 11, 13, 16, 18}, {8, 11, 14, 16, 18}, {8, 12, 14, 16, 18},
          {9, 11, 13, 15, 17}, {9, 11, 13, 15, 18}, {9, 11, 13, 16, 18},
          {9, 11, 14, 16, 18}, {9, 12, 14, 16, 18}, {10, 12, 14, 16, 18}}

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In[12]:= Length[%]
Out[12]= 2002
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B) Symbolic solution

The solution provided by the method of enumerating all possible cases was shown to be $\binom{14}{5}$. Why is this? Consider the following symbolic solution.

Place fourteen symbols or objects in a row. For example, fourteen letters:

a b c d e f g h i j k l m n

Now choose five from the fourteen (coloured green):

a b c d e f g h i j k l m n

Now insert five identical symbols (but different to the fourteen) to the immediate left (or equivalently, to the right) of the chosen five from the fourteen:

□ a b c □ d □ e f g h i □ j k l m □ n

Note that there are now nineteen symbols instead of eighteen. This is to ensure that the symbols at each end are treated in the same way.

Remove the last of the original fourteen symbols and then number the symbols from 1 to 18:

□	a	b	c	□	d	□	e	f	g	h	i	□	j	k	l	m	□
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18

The five chosen numbers from eighteen are those that correspond to the five inserted symbols ie 1, 5, 7, 13 and 18.

The method initially chooses five numbers from fourteen and finishes by obtaining five numbers from eighteen such that any two of the numbers differ by at least two.